出版前言

2011年12月11日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰100周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共10卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Contents

Section 1

The Buckling of Spherical Shells by External Pressure

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Bending Energy / unit area = $\frac{D}{2}\left[(k_1+k_2)^2 - 2(1-\mu)(k_1 k_2 - \tau^2)\right]$

Extensional Energy / unit area = $\frac{Et}{2}\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\mu)(\varepsilon_1 \varepsilon_2 - \frac{1}{4}\gamma^2)\right]$

Classical Value = $\phi = \frac{\sigma}{E\left(\frac{t}{R}\right)} = \frac{1}{\sqrt{3(1-\mu^2)}} \cong 0.6$

The original element

$$= \frac{dr}{\cos\Theta_0}$$

The deflected element

$$= \frac{dr}{\cos\Theta}$$

(1/4)

$$\varepsilon = \frac{\frac{dr}{\cos\Theta} - \frac{dr}{\cos\Theta_0}}{\frac{dr}{\cos\Theta_0}} = \frac{\cos\Theta_0}{\cos\Theta} - 1$$

The extensional energy

$$\frac{Et}{2}\int_0^a \left(\frac{\cos\Theta_0}{\cos\Theta} - 1\right)^2 2\pi r \frac{dr}{\cos\Theta_0}$$

$$= \frac{Et}{2}\int_0^\beta \left(\frac{\cos\Theta_0}{\cos\Theta} - 1\right)^2 2\pi R \sin\alpha \; R\, d\alpha$$

$$= \frac{ER^3}{2}\left(\frac{t}{R}\right)\cdot 2\pi \int_0^\beta \left(\frac{\cos\Theta_0}{\cos\Theta} - 1\right)^2 \sin\alpha\, d\alpha$$

The curvature in meridional direction $= \frac{d\Theta}{dS} = \frac{d\Theta}{d\alpha} \Big/ \frac{dS}{d\alpha}$

$$dS = \frac{dr}{\cos\Theta} = \frac{d[R\sin\alpha]}{\cos\Theta} = \frac{R\cos\alpha\, d\alpha}{\cos\Theta}$$

Change in curvature $= \dfrac{\frac{d\Theta}{d\alpha}}{\frac{R\cos\alpha}{\cos\Theta}} - \frac{1}{R} = \frac{1}{R}\left[\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha} - 1\right]$

Change in curvature in other direction

$$= \left(\frac{1}{\frac{R \sin\alpha}{\sin\Theta}} \right) - \frac{1}{R} = \frac{1}{R} \left[\frac{\sin\Theta}{\sin\alpha} - 1 \right]$$

The bending energy

$$= \frac{Et^3}{24} \int_0^\beta 2\pi R^2 \sin\alpha \, d\alpha \, \frac{1}{R^2} \left[\left(\frac{\cos\Theta}{\cos\alpha} \frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\sin\Theta}{\sin\alpha} - 1 \right)^2 \right]$$

$$= \frac{R^3 E}{2} \frac{\left(\frac{t}{R}\right)^3}{12} (2\pi) \int_0^\beta \sin\alpha \left[\left(\frac{\cos\Theta}{\cos\alpha} \frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\sin\Theta}{\sin\alpha} - 1 \right)^2 \right] d\alpha$$

The potential energy $\; p \cdot \int_0^a 2\pi r z \, dr$

$$= p \left[\left[2\pi \frac{r^2}{2} z \right]_0^a - \int_0^a 2\pi \frac{r^2}{2} \frac{dz}{dr} dr \right]$$

$$= - p \pi \int_0^\alpha R^2 \sin^2\alpha \tan\Theta \, R \cos\alpha \, d\alpha$$

$$= - p R^3 \pi \int_0^\alpha \sin^2\alpha \tan\Theta \cos\alpha \, d\alpha$$

$$E\left(\frac{t}{R}\right)\int_0^\beta\left(\frac{\cos\alpha}{\cos\Theta}-1\right)^2\sin\alpha\,d\alpha+\frac{E\left(\frac{t}{R}\right)^3}{12}\int_0^\beta\left\{\left(\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}-1\right)^2+\left(\frac{\sin\Theta}{\sin\alpha}-1\right)^2\right\}\sin\alpha\,d\alpha \quad (85)$$

$$+\ p\int_0^\beta \sin^2\alpha\cos\alpha\tan\Theta\,d\alpha$$

The differential equation

$$E\left(\frac{t}{R}\right)\left[2\sin\alpha\left(\frac{\cos\alpha}{\cos\Theta}-1\right)\frac{\cos\alpha}{\cos^2\Theta}\sin\Theta\right]$$

$$+\frac{E\left(\frac{t}{R}\right)^3}{12}\left[2\sin\alpha\left\{-\left(\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}-1\right)\frac{\sin\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}+\left(\frac{\sin\Theta}{\sin\alpha}-1\right)\frac{\cos\Theta}{\sin\alpha}\right\}\right.$$

$$\left.-\frac{d}{d\alpha}\left\{2\sin\alpha\left(\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}-1\right)\frac{\cos\Theta}{\cos\alpha}\right\}\right]$$

$$+\ p\left[\sin^2\alpha\cos\alpha\sec^2\Theta\right]=0$$

$$2E\left(\frac{t}{R}\right)\left[\left(\frac{\cos\alpha}{\cos\Theta}-1\right)\frac{\sin\alpha\cos\alpha}{\cos\Theta}\tan\Theta\right]$$

$$+\frac{E\left(\frac{t}{R}\right)^3}{6}\left[\sin\alpha\left\{\left(\frac{\sin\Theta}{\sin\alpha}-1\right)\frac{\cos\Theta}{\sin\alpha}-\left(\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}-1\right)\frac{\sin\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}\right\}\right.$$

$$-\left\{\sec^2\alpha\cos\Theta\left(\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}-1\right)-\tan\alpha\sin\Theta\left(\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}-1\right)\frac{d\Theta}{d\alpha}\right.$$

$$\left.\left.+\tan\alpha\cos\theta\left(\frac{-\sin\Theta}{\cos\alpha}\left(\frac{d\Theta}{d\alpha}\right)^2+\frac{\cos\Theta\tan\alpha}{\cos\alpha}\left(\frac{d\Theta}{d\alpha}\right)+\frac{\cos\Theta}{\cos\alpha}\frac{d^2\Theta}{d\alpha^2}\right)\right\}\right]$$

$$+\ p\left[\sin^2\alpha\cos\alpha\sec^2\Theta\right]=0.$$

$$2E\left(\frac{t}{R}\right)\left[\frac{\sin\alpha\cos\alpha}{\cos\Theta}\tan\Theta\left(\frac{\cos\alpha}{\cos\Theta}-1\right)\right]$$

$$+\frac{E\left(\frac{t}{R}\right)^3}{6}\left[\cos\Theta\left(\frac{\sin\Theta}{\sin\alpha}+\tan^2\alpha\right)-\frac{\cos^2\Theta}{\cos\alpha}\left(2\tan^2\alpha+1\right)\frac{d\Theta}{d\alpha}+\frac{\sin\Theta\cos\Theta\tan\alpha}{\cos\alpha}\left(\frac{d\Theta}{d\alpha}\right)^2\right.$$

$$\left.-\frac{\cos^2\Theta\tan\alpha}{\cos\alpha}\frac{d^2\Theta}{d\alpha^2}\right]+\sin^2\alpha\cos\alpha\sec^2\Theta\, p=0.$$

The other boundary condition: if the support is a hinge support is

$$\tan\alpha\cos\Theta\left(\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}-1\right)=0 \quad \text{at} \quad \alpha=\beta$$

or

$$\frac{\cos\Theta}{\cos\alpha}\frac{d\Theta}{d\alpha}=1 \quad \text{at} \quad \alpha=\beta$$

[no change in curvature]

the other boundary condition is

$\Theta=0$ at $\alpha=0$

If we neglect the bending energy, then

$$2E\frac{t}{R}\left[\sin\Theta\left(\frac{\cos\alpha}{\cos\Theta}-1\right)\right]+\sin\alpha\, p=0.$$

Let Θ_1 be the value of Θ at $\alpha=\beta$.

$$2E\frac{t}{R}\left[\sin\Theta_1\left(\frac{\cos\beta}{\cos\Theta_1}-1\right)\right]+\sin\beta\, p=0.$$

~~$\left(E\frac{t}{R}\right)p = -\left(\frac{\cos\beta}{\cos\Theta_1}-1\right)\frac{\sin\Theta_1}{\sin\beta}$~~

$$\frac{p}{E\left(\frac{t}{R}\right)}=-2\frac{\sin\Theta_1}{\sin\beta}\left(\frac{\cos\beta}{\cos\Theta_1}-1\right)$$

$\frac{\partial p}{\partial\Theta_1}=0$ gives $\quad \cos\beta=\cos^3\Theta_1$;

$$\therefore \quad \frac{p}{E\left(\frac{t}{R}\right)} = -2\frac{\left[1-(\cos\beta)^{2/3}\right]^{\frac{1}{2}}}{\sin\beta}\left[(\cos\beta)^{2/3}-1\right] \qquad 187)$$

$$= 2\,\frac{\left[1-(\cos\beta)^{2/3}\right]^{3/2}}{\sin\beta}$$

When β is small, $1-(\cos\beta)^{2/3} = \frac{1}{3}\beta^2$

$$\frac{p}{E\left(\frac{t}{R}\right)} \approx 2\frac{1}{(3)^{3/2}}\beta^2 = 0.385\,\beta^2$$

When $\beta = 0.1$ $\quad \frac{p}{E\left(\frac{t}{R}\right)} \approx 0.00385 \quad\quad p \approx 0.00385\,E\left(\frac{t}{R}\right)$

The critical stress $= \frac{pr}{2t} = 0.001925\,E$

Experiments $= \frac{0.25\times 0.60}{900}E = 0.000167\,E$ for $\frac{R}{t} = 900$.

If $\beta = 0.0294$, the values will agree.

The necessary radius to make a 1.5" radius dent will be

51"

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from the relation

$$\sin\Theta\left(\frac{\cos\alpha}{\cos\Theta}-1\right)+\frac{1}{2}\sin\alpha\left(\frac{p}{E\frac{t}{R}}\right)=0 \qquad \left(\frac{p}{E\frac{t}{R}}\right) = k$$

$$\sin\Theta\cos\alpha - \sin\Theta\cos\Theta + \frac{1}{2}k\cdot\sin\alpha\cos\Theta = 0.$$

$$\left(\cos\Theta\cos\alpha - \cos2\Theta - \frac{1}{2}k\cdot\sin\alpha\sin\Theta\right)\frac{d\Theta}{d\alpha} - \sin\Theta\sin\alpha + \frac{k}{2}\cdot\cos\alpha\cos\Theta = 0.$$

$$\frac{d\Theta}{d\alpha} = \frac{\sin\Theta\sin\alpha - \frac{1}{2}k\cos\alpha\cos\Theta}{\cos\Theta\cos\alpha - \cos2\Theta - \frac{1}{2}k\sin\alpha\sin\Theta}$$

If Θ, α are small, then

$$\frac{1}{2}\Theta(\Theta^2-\alpha^2)+\frac{1}{2}k\alpha=0$$

or $$\Theta^3-\alpha^2\Theta+k\alpha=0$$

$$\therefore \quad \Theta=\left(-\frac{k\alpha}{2}+\sqrt{\frac{k^2\alpha^2}{4}-\frac{\alpha^6}{27}}\right)^{\frac{1}{3}}+\left(-\frac{k\alpha}{2}-\sqrt{\frac{k^2\alpha^2}{4}-\frac{\alpha^6}{27}}\right)^{\frac{1}{3}}$$

$$=\alpha^{\frac{1}{3}}\left[\left(-\frac{k}{2}+\sqrt{\frac{k^2}{4}-\frac{\alpha^4}{27}}\right)^{\frac{1}{3}}+\left(-\frac{k}{2}-\sqrt{\frac{k^2}{4}-\frac{\alpha^4}{27}}\right)^{\frac{1}{3}}\right]$$

etc.

$$\left(\frac{\cos\alpha}{\cos\Theta}-1\right)=-\frac{k}{2}\frac{\alpha}{\Theta}$$

$$\frac{d\Theta}{d\alpha}=\frac{2\Theta\alpha-k}{3\Theta^2-\alpha^2-k\Theta\alpha}$$

If the Θ, α are of same order and are small, the differential equation can be written as

$$E\left(\frac{t}{R}\right)\alpha\Theta(\Theta^2-\alpha^2)+\alpha^2 p+\frac{E\left(\frac{t}{R}\right)^3}{6}\left\{\left(\frac{\Theta}{\alpha}+\alpha^2\right)-(2\alpha^2+1)\frac{d\Theta}{d\alpha}+\alpha\Theta\left(\frac{d\Theta}{d\alpha}\right)^2-\alpha\frac{d^2\Theta}{d\alpha^2}\right\}=0$$

or approximately

$$\alpha\Theta(\Theta^2-\alpha^2)+k_1\alpha^2+k_2\left\{\frac{\Theta}{\alpha}-\frac{d\Theta}{d\alpha}-\alpha\frac{d^2\Theta}{d\alpha^2}\right\}=0.$$

where $k_1=\frac{p}{\left(E\frac{t}{R}\right)}$, $k_2=\frac{\left(\frac{t}{R}\right)^2}{6}$

If α, Θ are small, the total energy can be written as

$$E\left(\frac{t}{R}\right)\int_0^\beta \frac{1}{4}(\Theta^2-\alpha^2)^2\alpha\,d\alpha + \frac{E\left(\frac{t}{R}\right)^3}{12}\int_0^\beta\left\{\left(\frac{d\Theta}{d\alpha}-1\right)^2+\left(\frac{\Theta}{\alpha}-1\right)^2\right\}\alpha\,d\alpha + p\int_0^\beta \alpha^2\Theta\,d\alpha$$

Let $\Theta = C\alpha$, then we have

$$E\left(\frac{t}{R}\right)\frac{1}{4}\left(C^2-1\right)^2\int_0^\beta \alpha^5\,d\alpha + \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)^2\int_0^\beta 2\alpha\,d\alpha + pC\int_0^\beta \alpha^3\,d\alpha$$

$$= E\left(\frac{t}{R}\right)\frac{1}{24}\left(C^2-1\right)^2\beta^6 + \frac{E\left(\frac{t}{R}\right)^3}{12}\left(C-1\right)^2\beta^2 + \frac{pC}{4}\beta^4$$

Differentiate with respect to C,

$$\frac{E\left(\frac{t}{R}\right)}{6}\,C\left(C^2-1\right)\beta^2 + \frac{E\left(\frac{t}{R}\right)^3}{6}\frac{(C-1)}{\beta^2} + \frac{p}{4} = 0.$$

Or

$$p = \frac{2}{3}E\left(\frac{t}{R}\right)\left\{C\left(1-C^2\right)\beta^2 + \left(\frac{t}{R}\right)^2\frac{(1-C)}{\beta^2}\right\}$$

Minimizing with respect to β^2,

$$C\left(1-C^2\right)\beta^2 = \left(\frac{t}{R}\right)^2\frac{(1-C)}{\beta^2}$$

$$\text{or}\qquad \beta^2 = \left(\frac{t}{R}\right)\sqrt{\frac{1-C}{C(1-C^2)}} = \left(\frac{t}{R}\right)\frac{1}{C^{\frac{1}{2}}(1+C)^{\frac{1}{2}}}$$

$$\therefore\quad p = \frac{2}{3}E\left(\frac{t}{R}\right)\left\{2\left(\frac{t}{R}\right)C^{\frac{1}{2}}(1-C)(1+C)^{\frac{1}{2}}\right\}$$

$$= \frac{4}{3}E\left(\frac{t}{R}\right)^2\left\{C^{\frac{1}{2}}(1+C)^{\frac{1}{2}}(1-C)\right\}$$

Minimizing with respect to C, $\frac{\partial}{\partial C}\left[C-C^2-C^3+C^4\right]=0$

$$\text{or}\qquad 1-2C-3C^2+4C^3=0$$

Divide through by $c-1$,

$$4c^2+c-1=0$$

$$c = \frac{1}{8}\left[-1 \pm \sqrt{1+16}\right]$$

$$= \frac{1}{8}\left[-1 \pm \sqrt{17}\right] = \begin{matrix} +0.3904 \\ -0.640 \end{matrix}$$

$$c^{\frac{1}{2}}(1+c)^{\frac{1}{2}}(1-c) = 0.3904^{\frac{1}{2}} \times 1.3904^{\frac{1}{2}} \times 0.6096 = 0.625 \times 1.179 \times 0.6096$$

$$= 0.450$$

$$p = 0.60 E \left(\frac{t}{R}\right)^2$$

$$\sigma = 0.30 E \left(\frac{t}{R}\right)$$

$$\beta^2 = \left(\frac{t}{R}\right)\frac{1}{0.625 \times 1.179}$$

$$= 1.357 \left(\frac{t}{R}\right)$$

or $\left(\frac{t}{R}\right) = \frac{1}{900}$, $\beta^2 = 0.001507$

$\beta = 0.0388$

Let us put $\Theta = C\left[\alpha + \left(\frac{1}{c}-1\right)\frac{\alpha^2}{\beta}\right]$, when $\alpha=\beta$ $\Theta = \beta$.

then
$$\Theta^2 - \alpha^2 = \alpha^2\left\{C^2\left[1+\left(\frac{1}{c}-1\right)\frac{\alpha}{\beta}\right]^2 - 1\right\}$$

$$\frac{d\Theta}{d\alpha} = C\left[1 + 2\left(\frac{1}{c}-1\right)\frac{\alpha}{\beta}\right]$$

Thus put this values to the integral for potential energy,

$$\frac{E\left(\frac{t}{R}\right)}{4}\int_0^\beta \left\{ C^2\left[1+\left(\frac{1}{C}-1\right)\left(\frac{\alpha}{\beta}\right)\right]^2-1\right\}^2 \alpha^5\, d\alpha + \frac{E\left(\frac{t}{R}\right)^3}{12}\int_0^\beta \left\{\left[(C-1)+2(1-C)\left(\frac{\alpha}{\beta}\right)\right]^2+\left[(C-1)+(1-C)\frac{\alpha}{\beta}\right]^2\right\} \alpha\, d\alpha$$

$$+ pC\int_0^\beta \left\{\alpha^3+\left(\frac{1}{C}-1\right)\frac{\alpha^4}{\beta}\right\} d\alpha$$

$$= \frac{E\left(\frac{t}{R}\right)}{4}\int_0^\beta \left\{ (C^2-1)^2+4C^2\left(\frac{1}{C}-1\right)(C^2-1)\left(\frac{\alpha}{\beta}\right)+2C^2(3C^2-1)\left(\frac{1}{C}-1\right)^2\left(\frac{\alpha}{\beta}\right)^2 \right.$$
$$\left. + 4C^4\left(\frac{1}{C}-1\right)^3\left(\frac{\alpha}{\beta}\right)^3+ C^4\left(\frac{1}{C}-1\right)^4\left(\frac{\alpha}{\beta}\right)^4 \right\} \alpha^5\, d\alpha$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)^2\int_0^\beta \left\{2-6\left(\frac{\alpha}{\beta}\right)+5\left(\frac{\alpha}{\beta}\right)^2\right\}\alpha\, d\alpha + pC\int_0^\beta\left\{\alpha^3+\left(\frac{1}{C}-1\right)\frac{\alpha^4}{\beta}\right\}d\alpha$$

$$= \frac{E\left(\frac{t}{R}\right)}{4}\left\{\frac{(C^2-1)^2}{6}+\frac{4}{7}C(1-C)(C^2-1)+\frac{1}{4}(3C^2-1)(1-C)^2+\frac{4}{9}C(1-C)^3+\frac{1}{10}(1-C)^4\right\}\beta^6$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)^2\frac{\beta^2}{4} + pC\left\{\frac{1}{4}+\frac{1}{5}\left(\frac{1}{C}-1\right)\right\}\beta^4$$

Minimizing with respect to C,

$$\frac{E\left(\frac{t}{R}\right)}{4}\left\{\frac{2}{3}C(C^2-1)-\frac{4}{7}(C-1)(4C^2+C-1)+\frac{1}{2}(C-1)(6C^2-3C-1)-\frac{4}{9}(C-1)(4C^2-5C+1)\right.$$
$$\left. +\frac{2}{5}(C-1)^3\right\}\beta^6$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}\frac{(C-1)\beta^2}{2} + p\left\{\frac{1}{4}-\frac{1}{5}\right\}\beta^4 = 0$$

$$p = 5E\left(\frac{t}{R}\right)\left[(1-C)\beta^2\left\{\frac{2}{3}C(C+1)-\frac{4}{7}(4C^2+C-1)+\frac{1}{2}(6C^2-3C-1)-\frac{4}{9}(4C^2-5C+1) +\frac{2}{5}(C-1)^2\right\} + \left(\frac{t}{R}\right)^2\frac{(1-C)}{6\beta^2}\right]$$

Minimizing with respect to β^2

$$\beta^2 = \frac{\left(\frac{t}{R}\right)}{\sqrt{6}}\left\{ \qquad \right\}^{-\frac{1}{2}}$$

$$\therefore\ p = \frac{10\left(\frac{t}{R}\right)^2 E}{\sqrt{6}}\left[(1-c)\left\{\tfrac{2}{3}c(c+1)-\tfrac{4}{7}(4c^2+c-1)+\tfrac{1}{2}(6c^2-3c-1)-\tfrac{4}{9}(4c^2-5c+1) + \tfrac{2}{5}(c-1)^2\right\}^{\frac{1}{2}}\right]$$

$$= \frac{5E\left(\frac{t}{R}\right)^2}{3\sqrt{105}}(1-c)\left\{420(c^2+c)-360(4c^2+c-1)+315(6c^2-3c-1) - 280(4c^2-5c+1)+252(c^2-2c+1)\right\}^{\frac{1}{2}}$$

$$= \frac{5}{3}\frac{E\left(\frac{t}{R}\right)^2}{\sqrt{105}}(1-c)\left\{2c^2+11c+17\right\}^{\frac{1}{2}}$$

Minimizing $(1-c)^2(2c^2+11c+17)$

$$-2(2c^2+11c+17)+(1-c)(4c+11)=0$$

or $4c^2+22c+34+4c^2+11c-4c-11=0$

or $8c^2+29c+23=0$

$$c = \frac{1}{16}\left[-29 \pm \sqrt{29^2-32\times 23}\right] = \frac{1}{16}\left[-29\pm\sqrt{105}\right]$$

$$= \frac{1}{16}\left[-29 \pm 10.25\right] = \begin{matrix}-1.171\\-2.452\end{matrix}$$

$$(2c^2+11c+17) = (1-c)(2c+5.5) = \begin{matrix}2.271\times 3.158 = 7.175\\ 3.452\times 0.596 = 2.056\end{matrix}$$

$$(1-c)(2c^2+11c+17)^{\frac{1}{2}} = \begin{matrix}2.271\times 2.678 = 6.08\\ 3.452\times 1.434 = 4.95\end{matrix}$$

$$p = \frac{5 \times 4.95}{3 \times 10.25} E\left(\frac{t}{R}\right)^2 = \underline{\underset{0.989}{0.804}} E\left(\frac{t}{R}\right)^2$$

$$\sigma = \underline{\underset{0.499}{0.402}} E\left(\frac{t}{R}\right)$$

$$\sigma = k \, E\left(\frac{t}{R}\right)$$

$C = 1,$ $\quad k = 0$

$C = 0.5$ $\quad k = \frac{5}{30.75} \frac{0.5 \times 4.796}{2} = 0.1951$

$C = 0$ $\quad k = \frac{5}{30.75} \frac{17^{\frac{1}{2}}}{2} = 0.335$

$C = -0.5,$ $\quad k = \frac{5}{30.75} \frac{1.5 \times 12^{\frac{1}{2}}}{2} = 0.422$

$C = -1.171$ $\quad k = 0.499$

$C = -1.5$ $\quad k = \frac{5}{30.75} \frac{2.5 \times 5^{\frac{1}{2}}}{2} = 0.455$

$C = -2.0$ $\quad k = \frac{5}{30.75} \frac{3.0 \times 3^{\frac{1}{2}}}{2} = 0.422$

$C = -2.452$ $\quad k = 0.402$

$C = -3$ $\quad k = \frac{5}{30.75} \, 2 \times 2^{\frac{1}{2}} = 0.460$

$C = -4$ $\quad k = \frac{5}{30.75} \, 2.5 \times 5^{\frac{1}{2}} = 0.909$

for free end, $p = \frac{4}{3} E(\frac{t}{R})^2 \ (c+c^2)^{\frac{1}{2}}(1-c)$

$$k = \frac{2}{3}(1-c)(c+c^2)^{\frac{1}{2}}$$

$C = 1$ $\quad k = 0$

$C = 0.1$ $\quad k = \frac{2}{3}\, 0.9 \quad 0.11^{\frac{1}{2}} = 0.1991$

$C = 0.2$ $\quad k = \frac{2}{3}\, 0.8 \quad 0.24^{\frac{1}{2}} = 0.2612$

$C = 0.3$ $\quad k = \frac{2}{3}\, 0.7 \quad 0.39^{\frac{1}{2}} = 0.2915$

$C = 0.3904$ $\quad k = 0.300$

$C = 0.50$ $\quad k = \frac{2}{3}\, 0.5 \quad 0.75^{\frac{1}{2}} = 0.2885$

$C = 0.60$ $\quad k = \frac{2}{3}\, 0.4 \quad 0.96^{\frac{1}{2}} = 0.2612$

$C = 0.8$ $\quad k = \frac{2}{3}\, 0.2 \quad 1.44^{\frac{1}{2}} = 0.1600$

$C = 0$ $\quad k = 0$

$C = -1$ $\quad k = 0$

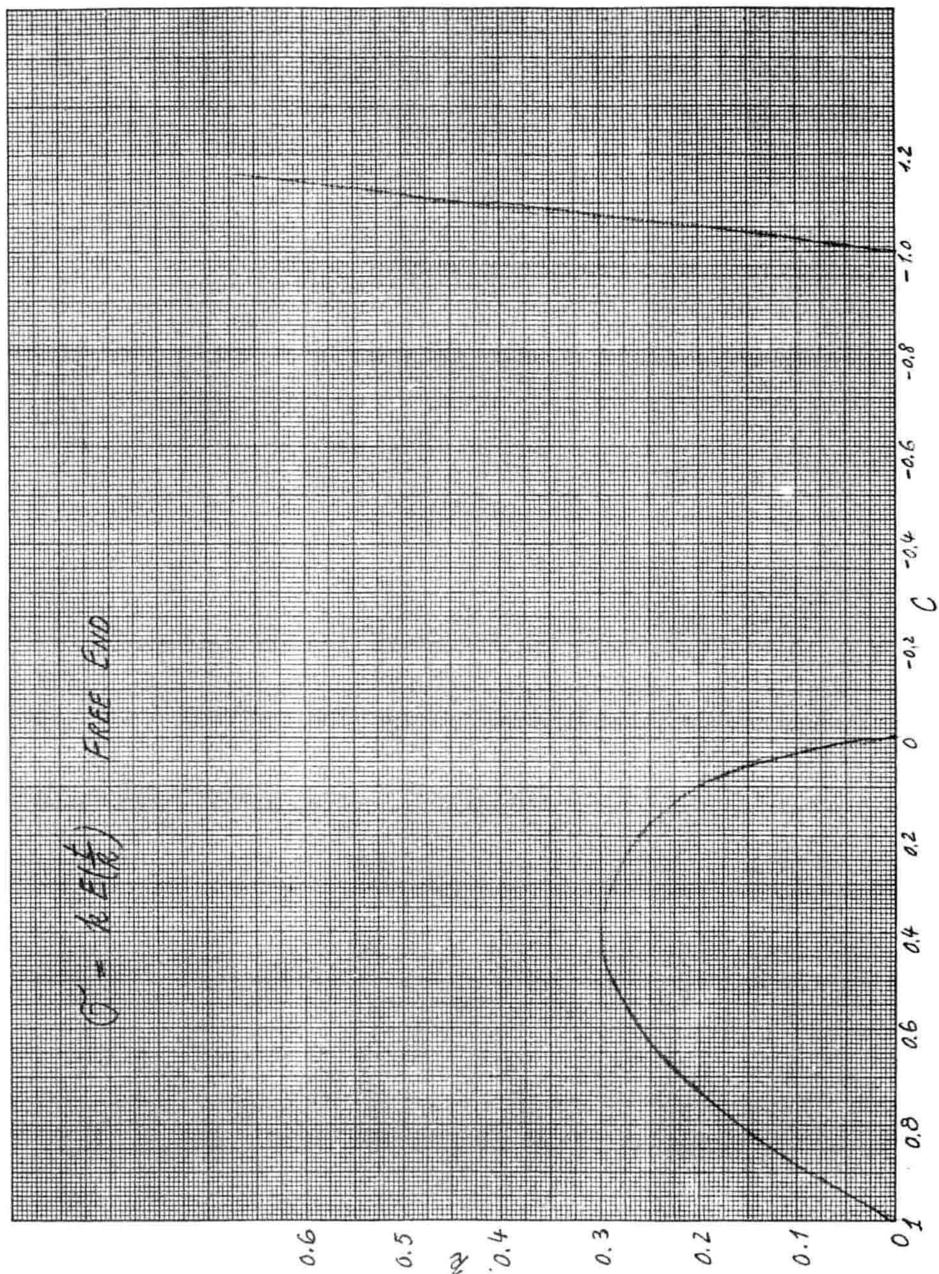
$C = -1.1$ $\quad k = \frac{2}{3}\, 2.1 \times 0.11^{\frac{1}{2}} = 0.465$

$C = -1.15$ $\quad k = \frac{2}{3}\, 2.15 \times 0.1725^{\frac{1}{2}} = 0.595$

(margin calculations:)

1.3225
1.15
0.1725

1.21
1.1
.11



σ = k E(t/R)
FREE END
KEUFFEL & ESSER CO., N. Y. NO. 359-14
Millimeters, 10th lines heavy.
MADE IN U.S.A.
C
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
1.2
0
0.1
0.2
0.3
0.4
0.5
0.6

We have the differential equation

$$\alpha\Theta(\Theta^2-\alpha^2)+k_1\alpha^2+k_2\left\{\frac{\Theta}{\alpha}-\frac{d\Theta}{d\alpha}-\alpha\frac{d^2\Theta}{d\alpha^2}\right\}=0$$

Divide through by β^4 and write $\frac{\alpha}{\beta}\sim\alpha,\ \frac{\Theta}{\beta}\sim\Theta,$

$$K_1=\frac{k_1}{\beta^2}=\frac{2\sigma}{E\beta^2},\quad K_2=\frac{\left(\frac{t}{R}\right)^2}{6\beta^4}$$

$$\alpha\Theta(\Theta^2-\alpha^2)+K_1\alpha^2+K_2\left\{\frac{\Theta}{\alpha}-\frac{d\Theta}{d\alpha}-\alpha\frac{d^2\Theta}{d\alpha^2}\right\}=0$$

Boundary condition for clamped end, $\Theta=0$ when $\alpha=0$

$\Theta=1$ when $\alpha=1$.

$$\alpha^2\frac{d^2\Theta}{d\alpha^2}+\alpha\frac{d\Theta}{d\alpha}-\Theta=\frac{1}{K_2}\alpha^2\Theta(\Theta^2-\alpha^2)+\frac{K_1}{K_2}\alpha^3$$

Taking the linear part

$$\alpha^2\frac{d^2\Theta}{d\alpha^2}+\alpha\frac{d\Theta}{d\alpha}-\Theta=\frac{K_1}{K_2}\alpha^3$$

The appropriate solution is $\Theta_0=C\alpha+\frac{K_1}{8K_2}\alpha^3$

$$\frac{1}{K_2}\alpha^2\Theta(\Theta^2-\alpha^2)=\frac{1}{K_2}\alpha^4\left(C\alpha+\frac{K_1}{8K_2}\alpha^3\right)\left[\left(C+\frac{K_1}{8K_2}\alpha^2\right)^2-1\right]$$

$$=\frac{1}{K_2}\alpha^5\left\{\left(C+\frac{K_1}{8K_2}\alpha^2\right)^3-\left(C+\frac{K_1}{8K_2}\alpha^2\right)\right\}$$

$$=\frac{1}{K_2}\alpha^5\left[C(C^2-1)+(3C^2-1)\frac{K_1}{8K_2}\alpha^2+3C\left(\frac{K_1}{8K_2}\right)^2\alpha^4+\left(\frac{K_1}{8K_2}\right)^3\alpha^6\right]$$

$$=\frac{1}{K_2}\left[C(C^2-1)\alpha^5+(3C^2-1)\frac{K_1}{8K_2}\alpha^7+3C\left(\frac{K_1}{8K_2}\right)^2\alpha^9+\left(\frac{K_1}{8K_2}\right)^3\alpha^{11}\right]$$

$$\Theta_1=C\alpha+\frac{K_1}{8K_2}\alpha^3+\frac{1}{K_2}\alpha^5\left[\frac{C(C^2-1)}{24}+(3C^2-1)\frac{K_1}{8K_2}\frac{\alpha^2}{48}+3C\left(\frac{K_1}{8K_2}\right)^2\frac{\alpha^4}{80}\right.$$
$$\left.+\left(\frac{K_1}{8K_2}\right)^3\frac{\alpha^6}{120}\right]$$

Write the original equation

$$\alpha^2 \frac{d^2\Theta}{d\alpha^2} + \alpha \frac{d\Theta}{d\alpha} - \Theta = C_1 \alpha^3 + C_2 \alpha^2 \Theta(\Theta^2 - \alpha^2)$$

where $\quad C_1 = \frac{K_1}{K_2}, \quad C_2 = \frac{1}{K_2}.$

Let $\quad \Theta = a_1 \alpha + a_2 \alpha + a_3 \alpha^3 + a_4 \alpha^4 + a_5 \alpha^5 + a_6 \alpha^6 + a_7 \alpha^7 + a_8 \alpha^8 + \cdots$

$$\Theta^2 - \alpha^2 = (a_1^2 - 1)\alpha^2 + 2a_1 a_2 \alpha^3 + (2a_1 a_3 + a_2^2)\alpha^4 + (2a_1 a_4 + 2a_2 a_3)\alpha^5$$
$$+ (2a_1 a_5 + 2a_2 a_4 + a_3^2)\alpha^6 + (2a_1 a_6 + 2a_2 a_5 + 2a_3 a_4)\alpha^7$$
$$+ (2a_1 a_7 + 2a_2 a_6 + 2a_3 a_5 + a_4^2)\alpha^8 + (2a_1 a_8 + 2a_2 a_7 + 2a_3 a_6 + 2a_4 a_5)\alpha^9 + \cdots$$

$$\Theta(\Theta^2 - \alpha^2) = a_1(a_1^2 - 1)\alpha^3 + [2a_1^2 a_2 + a_2(a_1^2 - 1)]\alpha^4 + [a_1(2a_1 a_3 + a_2^2) + 2a_1 a_2^2 + a_3(a_1^2 - 1)]\alpha^5$$
$$+ [a_1(2a_1 a_4 + 2a_2 a_3) + a_2(2a_1 a_3 + a_2^2) + 2a_1 a_2 a_3 + a_4(a_1^2 - 1)]\alpha^6 + \cdots$$

$$C_1 \alpha^3 + C_2 \alpha^2 \Theta(\Theta^2 - \alpha^2)$$

$$= C_1 \alpha^3 + 0 + C_2 a_1(a_1^2 - 1)\alpha^5 + C_2[2a_1^2 a_2 + a_2(a_1^2 - 1)]\alpha^6$$
$$+ C_2[a_1(2a_1 a_3 + a_2^2) + 2a_1 a_2^2 + a_3(a_1^2 - 1)]\alpha^7$$
$$+ C_2[a_1(2a_1 a_4 + 2a_2 a_3) + a_2(2a_1 a_3 + a_2^2) + 2a_1 a_2 a_3 + a_4(a_1^2 - 1)]\alpha^8 + \cdots$$

$$\left(\alpha^2 \frac{d^2}{d\alpha^2} + \alpha \frac{d}{d\alpha} - 1\right) a_n \alpha^n = (n^2 - 1) a_n \alpha^n$$

Thus
$$a_1 = ?$$
$$a_2 = 0$$
$$a_3 = \frac{C_1}{8}$$
$$a_4 = 0$$

$$a_5 = \frac{C_2}{24}\left[a_1(a_1^2-1)\right]$$

$$a_6 = 0$$

$$a_7 = \frac{C_2}{48}\left[\frac{a_1^2 C_1}{4} + \frac{C_1}{8}(a_1^2-1)\right] = \frac{C_2 C_1}{384}(3a_1^2-1)$$

$$a_8 = 0.$$

$$a_9 = \frac{C_2}{80}\left[a_1\left\{2a_1^2\frac{C_2}{24}(a_1^2-1) + \frac{C_1^2}{64}\right\} + \frac{C_1}{8}\left\{2a_1\frac{C_1}{8}\right\} + \frac{C_2}{24}a_1(a^2-1)^2\right]$$

$$= \frac{C_2}{80}\left[\frac{C_2}{24}a_1(3a_1^2-1)(a_1^2-1) + \frac{C_1^2}{64}\cdot 3a_1\right]$$

$$= \frac{C_2}{640}\left[\frac{C_2}{3}a_1(3a_1^2-1)(a_1^2-1) + \frac{3}{8}C_1^2 a_1\right].$$

The condition for a_1 is [<u>Free end</u>].

$$1 = a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 \cdots$$

$$\therefore\ 1 = a_1 + \frac{3}{8}C_1 + \frac{5}{24}C_2 a_1(a_1^2-1) + \frac{7}{384}C_1C_2(3a_1^2-1) + \frac{9}{640}C_2\left[\frac{C_2}{3}a_1(3a_1^2-1)(a_1^2-1) + \frac{3}{8}C_1^2 a_1\right]$$

$+ - - - - - - -$

This can be put into the form

$$A'C_1^2 + B'C_1 + D' = 0.$$

where $A' = \frac{27}{5120}a_1 C_2$

$B' = \frac{3}{8} + \frac{7}{384}(3a_1^2-1)C_2$

$C' = (a_1-1) + \frac{5}{24}a_1(a_1^2-1)C_2 + \frac{3}{640}a_1(3a_1^2-1)(a_1^2-1)C_2^2$

$$C_1 = \frac{\frac{2\sigma}{E\beta^2}}{\frac{\left(\frac{t}{R}\right)^2}{6\beta^4}} = 12\left(\frac{\sigma}{E\frac{t}{R}}\right)\left(\frac{\beta^2}{\frac{t}{R}}\right) = 12\left(\frac{\beta^2}{\frac{t}{R}}\right)s \qquad 200)$$

where $s = \frac{\sigma}{\left(E\frac{t}{R}\right)}$ $\qquad C_2 = 6\left(\frac{\beta^2}{\frac{t}{R}}\right)^2$

$$A'\,144\left(\frac{\beta^2}{\frac{t}{R}}\right)^2 s^2 + 12\left(\frac{\beta^2}{\frac{t}{R}}\right)B'\,s + D' = 0$$

$$As^2 + Bs + D = 0$$

$$A = \frac{27}{5120}a_1\, 6\left[\frac{\beta^2}{\left(\frac{t}{R}\right)}\right]^2 144\left[\frac{\beta^2}{\left(\frac{t}{R}\right)}\right]^2 = \frac{729}{160}\left\{\frac{\beta^2}{\left(\frac{t}{R}\right)}\right\}^4 a_1 = \frac{729}{160}\phi^4 a_1$$

$$B = 12\left(\frac{\beta^2}{\frac{t}{R}}\right)\left[\frac{3}{8} + \frac{42}{384}(3a_1^2-1)\left(\frac{\beta^2}{\left(\frac{t}{R}\right)}\right)^2\right] = \phi\left\{\frac{9}{4} + \frac{21}{16}(3a_1^2-1)\phi^2\right\}$$

$$D = (a_1-1) + \frac{5}{4}a_1(a_1^2-1)\phi^2 + \frac{27}{160}a_1(3a_1^2-1)(a_1^2-1)\phi^4$$

When $\phi = 1$, $\quad \beta = 0.0316$, $\quad \frac{t}{R} = \frac{1}{1000}$

$$A = \frac{729}{160}a_1$$

$$B = \frac{9}{4} + \frac{21}{16}(3a_1^2-1)$$

$$D = (a_1-1) + \frac{5}{4}a_1(a_1^2-1) + \frac{27}{160}a_1(3a_1^2-1)(a_1^2-1)$$

$$= (a_1-1)\left\{1 + \frac{5}{4}a_1(a_1+1) + \frac{27}{160}a_1(a_1+1)(3a_1^2-1)\right\}$$

(1) $a_1 = 1.0$, $\zeta = 0$

(2) $a_1 = 0.9$, $A = 4.10$ $B = 2.25 + \frac{21}{16} 1.43 = \begin{array}{r} 2.25 \\ 1.877 \\ \hline 4.127 \end{array}$

$$D = -0.1\left[1 + \frac{5}{4} 0.9 \times 1.9 + \frac{27}{160} 0.9 \times 1.9 \times 1.43\right]$$

$$= -0.1\left(1 + 2.136 + 0.4125\right) = -0.3549$$

$$\zeta = \frac{1}{8.20}\left[-4.127 \pm \sqrt{4.127^2 + 16.40 \times 0.3549}\right]$$

$$= \frac{1}{8.20}\left[-4.127 \pm 4.127 \times 1.1582\right] = +0.0798.$$

(3) $a_1 = 0.8$ $A = 3.644$, $B = 2.25 + \frac{21}{16} 0.92 = \begin{array}{r} 2.25 \\ 1.207 \\ \hline 3.457 \end{array}$

$$D = -0.2\left[1 + 0.8 \times 1.8\left(1.25 + \frac{27}{160} \times 0.92\right)\right]$$

$$= -0.2\left(1 + 2.024\right) = -0.6048$$

$$\zeta = \frac{1}{7.288}\left[-3.457 \pm \sqrt{3.457^2 + 14.576 \times 0.6048}\right]$$

$$= \frac{3.457}{7.288}\left[-1 + 1.318\right] = 0.151$$

(4) $a_1 = 0.7$ $A = 3.190$ $B = 2.25 + \frac{21}{16} \times 0.47 = 2.8665$

$$D = -0.3\left[1 + 0.7 \times 1.7\left(1.25 + \frac{27}{160} \times 0.47\right)\right] = -0.774$$

$$\zeta = \frac{1}{6.38}\left[-2.8665 \pm 2.8665 \times 1.483\right] = +0.217$$

(5) $a_1 = 0.6$ $A = 2.730$ $B = 2.25 + \frac{21}{16} \times 0.08 = 2.355$

$$D = -0.4\left[1 + 0.6 \times 1.6\left(1.25 + \frac{27}{160} \times 0.08\right)\right] = -0.885$$

$$\varsigma = \frac{2.355}{5.460} \times 0.655 = 0.2824$$

(6) $a_1 = 0.5$ $\qquad \varsigma = +0.359$

(7) $a_1 = 0$, $\quad A = 0$, $B = \frac{15}{16}$. $\quad D = -1$

$$\varsigma = \frac{16}{15} = 1.066$$

---

for clamped end

$$1 = a_1 + \frac{C_1}{8} + \frac{C_2}{24} a_1 (a_1^2 - 1) + \frac{C_1 C_2}{384} (3a_1^2 - 1) + \frac{C_2}{640} \left[\frac{C_2}{3} a_1 (a_1^2 - 1)(3a_1^2 - 1) + \frac{3}{8} C_1 a_1^2 \right]$$

$$A' = \frac{3}{5120} a_1 C_2$$

$$B' = \frac{1}{8} + \frac{1}{384} (3a_1^2 - 1) C_2$$

$$D' = (a_1 - 1) + \frac{1}{24} a_1 (a_1^2 - 1) C_2 + \frac{1}{1920} a_1 (a_1^2 - 1)(3a_1^2 - 1) C_2^2$$

$$A = \frac{81}{160} \phi^4 a_1$$

$$B = \phi \left\{ \frac{3}{4} + \frac{3}{16} (3a_1^2 - 1) \phi^2 \right\}$$

$$D = (a_1 - 1) + \frac{1}{4} a_1 (a_1^2 - 1) \phi^2 + \frac{3}{160} a_1 (3a_1^2 - 1)(a_1^2 - 1) \phi^4$$

$$\frac{81}{160}a_1\frac{\beta^8}{\left(\frac{t}{r}\right)^4}\zeta^2+\frac{\beta^2}{\left(\frac{t}{r}\right)}\left\{\frac{3}{4}+\frac{3}{16}(3a_1^2-1)\frac{\beta^4}{\left(\frac{t}{r}\right)^2}\right\}\zeta+\left\{(a_1-1)+\frac{1}{4}a_1(a_1^2-1)\frac{\beta^4}{\left(\frac{t}{r}\right)^2}+\frac{3}{160}a_1(3a_1^2-1)(a_1^2-1)\frac{\beta^8}{\left(\frac{t}{r}\right)^4}\right\}=0.$$

$$\frac{81}{40}a_1\frac{\beta^6}{\left(\frac{t}{r}\right)^4}\zeta^2+\frac{1}{\left(\frac{t}{r}\right)}\left\{\frac{3}{4}+\frac{9}{16}(3a_1^2-1)\frac{\beta^4}{\left(\frac{t}{r}\right)^2}\right\}\zeta+\left\{\frac{1}{2}a_1(a_1^2-1)\frac{\beta^2}{\left(\frac{t}{r}\right)^2}+\frac{3}{40}a_1(3a_1^2-1)(a_1^2-1)\frac{\beta^6}{\left(\frac{t}{r}\right)^4}\right\}=0.$$

$$\frac{81}{40}a_1\phi^4\zeta^2+\phi\left\{3+\frac{3}{4}(3a_1^2-1)\phi^2\right\}\zeta+\left\{4(a_1-1)+a_1(a_1^2-1)\phi^2+\frac{3}{40}a_1(a_1^2-1)(3a_1^2-1)\phi^4\right\}=0$$

$$\frac{81}{40}a_1\phi^4\zeta^2+\phi\left\{\frac{3}{4}+\frac{9}{16}(3a_1^2-1)\phi^2\right\}\zeta+\left\{\frac{1}{2}a_1(a_1^2-1)\phi^2+\frac{3}{40}a_1(a_1^2-1)(3a_1^2-1)\phi^4\right\}=0.$$

$$\phi\left\{\frac{9}{4}+\frac{3}{16}(3a_1^2-1)\phi^2\right\}\zeta+\left\{4(a_1-1)+\frac{1}{2}a_1(a_1^2-1)\phi^2\right\}=0.$$

$$\zeta=-\frac{4(a_1-1)+\frac{1}{2}a_1(a_1^2-1)\phi^2}{\phi\left\{\frac{9}{4}+\frac{3}{16}(3a_1^2-1)\phi^2\right\}}$$

Use energy method, $\Theta = C\left[\alpha + \left(\frac{1}{C}-1\right)\frac{\alpha^3}{\beta^2}\right]$

$$\Theta^2 - \alpha^2 = \alpha^2\left\{C^2\left[1+\left(\frac{1}{C}-1\right)\left(\frac{\alpha}{\beta}\right)^2\right]^2 - 1\right\}$$

$$= \alpha^2\left\{(C^2-1) + 2C^2\left(\frac{1}{C}-1\right)\left(\frac{\alpha}{\beta}\right)^2 + C^2\left(\frac{1}{C}-1\right)^2\left(\frac{\alpha}{\beta}\right)^4\right\}$$

$$\frac{d\Theta}{d\alpha} = C\left[1+3\left(\frac{1}{C}-1\right)\left(\frac{\alpha}{\beta}\right)^2\right]$$

$$\frac{d\Theta}{d\alpha}-1 = (C-1)+3C\left(\frac{1}{C}-1\right)\left(\frac{\alpha}{\beta}\right)^2 = (C-1)\left\{1-3\left(\frac{\alpha}{\beta}\right)^2\right\}$$

$$\frac{\Theta}{\alpha} - 1 = (C-1)\left\{1-\left(\frac{\alpha}{\beta}\right)^2\right\}$$

The total potential energy

$$\frac{E\left(\frac{t}{R}\right)}{4}\int_0^\beta\left\{(C^2-1)^2 + 4C^2\left(\frac{1}{C}-1\right)(C^2-1)\left(\frac{\alpha}{\beta}\right)^2 + 2C^2(3C^2-1)\left(\frac{1}{C}-1\right)^2\left(\frac{\alpha}{\beta}\right)^4 + 4C^4\left(\frac{1}{C}-1\right)^3\left(\frac{\alpha}{\beta}\right)^6 + C^4\left(\frac{1}{C}-1\right)^4\left(\frac{\alpha}{\beta}\right)^8\right\}\alpha^5\,d\alpha$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)^2\int_0^\beta\left\{2-8\left(\frac{\alpha}{\beta}\right)^2+10\left(\frac{\alpha}{\beta}\right)^4\right\}\alpha\,d\alpha + pC\int_0^\beta\left\{\alpha^3+\left(\frac{1}{C}-1\right)\frac{\alpha^5}{\beta^2}\right\}d\alpha$$

$$= \frac{E\frac{t}{R}}{4}\left\{\frac{(C^2-1)^2}{6}+\frac{1}{2}C(1-C)(C^2-1)+\frac{1}{5}(3C^2-1)(1-C)^2+\frac{1}{3}C(1-C)^3+\frac{1}{14}(1-C)^4\right\}\beta^6$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)^2\frac{2}{3}\beta^2 + pC\left\{\frac{1}{4}+\frac{1}{6}\left(\frac{1}{C}-1\right)\right\}\beta^4$$

Minimizing with respect to C

$$\frac{E\left(\frac{t}{R}\right)}{4}\left\{\frac{2}{3}C(C^2-1) - \frac{1}{2}(C-1)(4C^2+C-1) + \frac{2}{5}(C-1)(6C^2-3C-1) - \frac{1}{3}(C-1)(4C^2-5C+1) + \frac{2}{7}(C-1)^3\right\}\beta^6$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)\frac{4}{3}\beta^2 + \frac{1}{12}p\beta^4 = 0$$

205)

$$p = 3E\left(\frac{t}{R}\right)\left[(1-c)\beta^2\left\{\frac{2}{3}c(c+1)-\frac{1}{2}(4c^2+c-1)+\frac{2}{5}(6c^2-3c-1)-\frac{1}{3}(4c^2-5c+1)+\frac{2}{7}(c-1)^2\right\}+\left(\frac{t}{R}\right)^2\frac{4}{9}(1-c)\frac{1}{\beta^2}\right]$$

Minimizing with respect to β^2

$$\beta^2 = \frac{2}{3}\left(\frac{t}{R}\right)\left\{\quad\right\}^{-\frac{1}{2}}$$

$$p = 4E\left(\frac{t}{R}\right)^2\left[(1-c)\left\{\frac{2}{3}c(c+1)-\frac{1}{2}(4c^2+c-1)+\frac{2}{5}(6c^2-3c-1)-\frac{1}{3}(4c^2-5c+1)+\frac{2}{7}(c-1)^2\right\}^{\frac{1}{2}}\right]$$

$$= \frac{4}{\sqrt{210}}E\left(\frac{t}{R}\right)^2(1-c)\left\{140(c^2+c)-105(4c^2+c-1)+84(6c^2-3c-1)-70(4c^2-5c+1)+60(c-1)^2\right\}^{\frac{1}{2}}$$

$$= \frac{4}{\sqrt{210}}E\left(\frac{t}{R}\right)^2(1-c)\left(4c^2+13c+11\right)^{\frac{1}{2}}$$

Minimizing $(1-c)^2(4c^2+13c+11)$

$$-2(4c^2+13c+11)+(1-c)(8c+13)=0$$

or

$$16c^2+31c+9=0$$

$$c = \frac{1}{32}\left[-31 \pm \sqrt{961-36\times 16}\right] = \frac{1}{32}\left[-31 \pm 19.62\right] = \begin{matrix}-0.3556 \\ -1.582\end{matrix}$$

$\xcancel{-0.3560}$

$\xcancel{-1.581}$

$$(4c^2+13c+11) = \frac{(1-c)(8c+13)}{2} = (1-c)(4c+6.5) = \begin{matrix} 1.356 \times 5.078 \\ 2.582 \times 0.1725 \end{matrix} \quad 206)$$

$$= \begin{matrix} 6.886 \\ 0.4454 \end{matrix}$$

$$K = (1-c)(4c^2+13c+11)^{\frac{1}{2}} = \begin{matrix} 1.356 \times 2.624 = 3.558 \\ 2.582 \times 0.6674 = 1.723 \end{matrix}$$

$$p = \frac{4}{14.49} K E \left(\frac{t}{R}\right)^2$$

$$\boxed{\sigma = \frac{2}{14.49} K E\left(\frac{t}{R}\right) = E\left(\frac{t}{R}\right) \begin{matrix} 0.4911 \\ 0.2378 \end{matrix}}$$

Ja!!!

$$\beta^2 = \frac{2 \times 14.49}{3 \times 0.674} \left(\frac{t}{R}\right) = \frac{2 \times 14.49}{2700 \times 0.674} = 0.01592$$

$$\boxed{\beta = 0.126 \text{ Radians} = \underline{\underline{7.2^\circ}}}$$

$$\frac{\sigma_a}{E\left(\frac{t}{R}\right)} = \frac{2}{\sqrt{210}} (1-c)(4c^2+13c+11)^{\frac{1}{2}}$$

$$= \frac{2}{14.49}$$

1756249
3512498
$\bar{1}.6487502$
$\bar{1}.8243751$

for $\Theta = C\alpha\left[1 + \left(\frac{1}{C} - 1\right)\left(\frac{\alpha}{\beta}\right)^2\right]$ (206a)

for p. 206

$\left(\frac{\delta}{t}\right)$	① C	② $4C^2$	③ $13C$	④ $4C^2+13C+11$	⑤ $(1-C)$	⑥ ④$^{\frac{1}{2}}$	⑦ ϕ
0	1	4	13	28	0	5.292	0
0.0985	0.8	2.56	10.4	23.96	0.2	4.900	0.1353
0.2148	0.6	1.44	7.8	20.24	0.4	4.500	0.2484
0.3530	0.4	0.64	5.2	16.84	0.6	4.105	0.3400
0.520	0.2	0.16	2.6	13.76	0.8	3.710	0.4097
0.727	0	0	0	11	1.0	3.317	0.4581
0.990	−0.2	0.16	−2.6	8.56	1.2	2.926	0.4846
1.333	−0.4	0.64	−5.2	6.44	1.4	2.538	0.4904
1.793	−0.6	1.44	−7.8	4.64	1.6	2.154	0.4757
2.44	−0.8	2.56	−10.4	3.16	1.8	1.778	0.4417
3.42	−1.0	4	−13	2.00	2.0	1.414	0.3903
4.93	−1.2	5.76	−15.6	1.16	2.2	1.077	0.3270
7.25	−1.4	7.84	−18.2	0.64	2.4	0.80	0.2650
9.47	−1.6	10.24	−20.8	0.44	2.6	0.6633	0.2380
	−1.8	12.96	−23.4	0.56	2.8	0.7483	0.2892
	−2.0	16.00	−26.0	1.00	3.0	1.0000	0.4141
	−2.2	19.36	−28.6	1.76	3.2	1.327	0.5861

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6

$$z_0 + \int_0^\beta \frac{dz}{dr}\,dr = 0$$

$$z_0 = -\int_0^\beta \frac{dz}{dr}\,dr = +R\int_0^\beta \tan\Theta \cos\alpha\, d\alpha$$

$$\cong +R\int_0^\beta \Theta\, d\alpha$$

$$\Theta = C\left[\alpha + \left(\frac{1}{C}-1\right)\frac{\alpha^3}{\beta^2}\right]$$

$$z_0 = +R\int_0^\beta \left[C\alpha + (1-C)\frac{\alpha^3}{\beta^2}\right]d\alpha$$

$$= +R\left[C\frac{\beta^2}{2} + \frac{1-C}{4}\beta^2\right]$$

$$= +R\left[\frac{1+C}{4}\right]\beta^2$$

Original $(z_0)_{ori} = R[1-\cos\beta] \cong R\frac{\beta^2}{2}$

$$\delta = R\beta^2\left[\frac{1}{2} - \frac{1+C}{4}\right] = R\beta^2\,\frac{(1-C)}{4} = z_{0_{ori}} - z_0$$

$$\frac{\delta}{R} = \frac{(1-C)}{4}\,\frac{2}{3}\left(\frac{t}{R}\right)\left\{\frac{2}{3}C(C+1) - \frac{1}{2}(4C^2+C-1) + \frac{2}{5}(6C^2-3C-1) - \frac{1}{3}(4C^2-5C+1) + \frac{2}{7}(C-1)^2\right\}^{-\frac{1}{2}}$$

But $\frac{1}{2}\left\{\frac{\sigma}{E\left(\frac{t}{R}\right)}\right\}\frac{1}{1-C} = \{\ \}^{\frac{1}{2}}$

$$\frac{\delta}{R} = \left(\frac{t}{R}\right)\cdot\frac{(1-C)}{4}\,\frac{2}{3}\,\frac{2(1-C)}{\phi} = \left(\frac{t}{R}\right)\frac{(1-C)(1-C)}{3\phi} = \frac{t}{R}\,\frac{(1-C)^2}{3\phi}$$

Put $\Theta = c_1\alpha\left[1+\left(\frac{1}{c_1}-1\right)\left(\frac{\alpha}{\beta}\right)^2\right] + c_2\alpha\left[1+\left(\frac{1}{c_2}-1\right)\left(\frac{\alpha}{\beta}\right)^4\right]$

$$= \alpha\left\{(c_1+c_2) + (1-c_1)\left(\frac{\alpha}{\beta}\right)^2 + (1-c_2)\left(\frac{\alpha}{\beta}\right)^4\right\}$$

$$\Theta^2-\alpha^2 = \alpha^2\left\{[(c_1+c_2)^2-1] + 2(1-c_1)(c_1+c_2)\left(\frac{\alpha}{\beta}\right)^2 + [(1-c_1)^2+2(c_1+c_2)(1-c_2)]\left(\frac{\alpha}{\beta}\right)^4 + 2(1-c_1)(1-c_2)\left(\frac{\alpha}{\beta}\right)^6 + (1-c_2)^2\left(\frac{\alpha}{\beta}\right)^8\right\}$$

$$\{\Theta^2-\alpha^2\}^2 = \alpha^4\Big\{[(c_1+c_2)^2-1]^2 + 4(1-c_1)(c_1+c_2)[(c_1+c_2)^2-1]\left(\frac{\alpha}{\beta}\right)^2$$

$$+ \{4(1-c_1)^2(c_1+c_2)^2 + 2[(c_1+c_2)^2-1][(1-c_1)^2+2(c_1+c_2)(1-c_2)]\}\left(\frac{\alpha}{\beta}\right)^4$$

$$+4\{[(c_1+c_2)^2-1](1-c_1)(1-c_2) + (1-c_1)(c_1+c_2)[(1-c_1)^2+2(c_1+c_2)(1-c_2)]\}\left(\frac{\alpha}{\beta}\right)^6$$

$$+\{2(1-c_2)^2[(c_1+c_2)^2-1] + 8(1-c_1)^2(1-c_2)(c_1+c_2) + [(1-c_1)^2+2(1-c_2)(c_1+c_2)]^2\}\left(\frac{\alpha}{\beta}\right)^8$$

$$+4\{(1-c_1)(1-c_2)^2(c_1+c_2) + (1-c_1)(1-c_2)[(1-c_1)^2+2(c_1+c_2)(1-c_2)]\}\left(\frac{\alpha}{\beta}\right)^{10}$$

$$+\{2(1-c_2)^2[(1-c_1)^2+2(1-c_2)(c_1+c_2)] + 4(1-c_1)^2(1-c_2)^2\}\left(\frac{\alpha}{\beta}\right)^{12}$$

$$+4(1-c_1)(1-c_2)^3\left(\frac{\alpha}{\beta}\right)^{14} + (1-c_2)^4\left(\frac{\alpha}{\beta}\right)^{16}\Big\}$$

$$\frac{d\Theta}{d\alpha}-1 = \left\{[(c_1+c_2)-1] + 3(1-c_1)\left(\frac{\alpha}{\beta}\right)^2 + 5(1-c_2)\left(\frac{\alpha}{\beta}\right)^4\right\}$$

$$\frac{\Theta}{\alpha}-1 = \left\{[(c_1+c_2)-1] + (1-c_1)\left(\frac{\alpha}{\beta}\right)^2 + (1-c_2)\left(\frac{\alpha}{\beta}\right)^4\right\}$$

$$\left(\frac{d\Theta}{d\alpha}-1\right)^2+\left(\frac{\Theta}{\alpha}-1\right)^2 = 2[(c_1+c_2)-1]^2 + 8(1-c_1)[(c_1+c_2)-1]\left(\frac{\alpha}{\beta}\right)^2$$

$$+\{10(1-c_1)^2+12(1-c_2)[(c_1+c_2)-1]\}\left(\frac{\alpha}{\beta}\right)^4 + 32(1-c_1)(1-c_2)\left(\frac{\alpha}{\beta}\right)^6$$

$$+26(1-c_2)^2\left(\frac{\alpha}{\beta}\right)^8$$

$$\int_0^\beta (\Theta^2 - \alpha^2)^2 \alpha\, d\alpha = \frac{1}{6}\left[(C_1+C_2)^2-1\right]^3 - \frac{1}{2}(C_1-1)(C_1+C_2)\left[(C_1+C_2)^2-1\right]$$

$$+ \frac{1}{5}\left\{ 2(C_1-1)^2(C_1+C_2)^2 + \left[(C_1+C_2)^2-1\right]\left[(C_1-1)^2 - 2(C_1+C_2)(C_2)\right]\right\}$$

$$+ \frac{1}{3}\left\{\left[(C_1+C_2)^2-1\right](C_1-1)(C_2) - (C_1-1)(C_1+C_2)\left[(C_1-1)^2 - 2(C_1+C_2)(C_2)\right]\right\}$$

$$+ \frac{1}{14}\left\{ 2(C_2)^2\left[(C_1+C_2)^2-1\right] - 8(C_1-1)^2(C_2)(C_1+C_2) + \left[(C_1-1)^2 - 2(C_1+C_2)(C_2)\right]^2\right\}$$

$$+ \frac{1}{4}\left\{ -(C_1-1)(C_2)^2(C_1+C_2) + (C_1-1)(C_2)\left[(C_1-1)^2 - 2(C_1+C_2)(C_2)\right]\right\}$$

$$+ \frac{1}{9}\left\{ (C_2)^2\left[(C_1-1)^2 - 2(C_1+C_2)(C_2)\right] + 2(C_1-1)^2(C_2)^2\right\}$$

$$+ \frac{1}{5}(C_1-1)(C_2)^3 + \frac{1}{22}(C_2)^4 \qquad \text{Multiplied by } \beta^6$$

$$\int_0^\beta \left\{\left(\frac{d\Theta}{d\alpha}-1\right)^2 + \left(\frac{\Theta}{\alpha}-1\right)^2\right\}\alpha\, d\alpha = \left[(C_1+C_2)-1\right]^2 - 2(C_1-1)\left[(C_1+C_2)-1\right]$$

$$+ \frac{1}{3}\left\{ 5(C_1-1)^2 - 6(C_2)\left[(C_1+C_2)-1\right]\right\} + 4(C_1-1)(C_2) + 2.6(C_2)^2$$

Multiplied by β^2

$$\int_0^\beta \alpha^2 \Theta\, d\alpha = \frac{1}{4}(C_1+C_2) - \frac{1}{6}(C_1-1) - \frac{1}{8}(C_2) \qquad \text{Multiply by } \beta^4$$

Put $\lambda = \dfrac{C_2}{C_1}$

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$$\int_0^\beta (\Theta^2-\alpha^2)\alpha\,d\alpha = \frac{1}{6}\left[C_1^2(1+\lambda)^2-1\right]^2 - \frac{1}{2}C_1(C_1-1)(1+\lambda)\left[C_1^2(1+\lambda)^2-1\right]$$

$$+\frac{1}{5}\left\{3C_1^2(C_1-1)^2(1+\lambda)^2 - 2C_1^3(C_1\lambda)(1+\lambda)^3 - (C_1-1)^2 + 2C_1(1+\lambda)(\lambda C_1)\right\}$$

$$+\frac{1}{3}\left\{3C_1^2(C_1-1)(1+\lambda)^2(\lambda C_1) - (C_1-1)(\lambda C_1) - C_1(C_1-1)^3(1+\lambda)\right\}$$

$$+\frac{1}{14}\left\{2(C_1\lambda)^2\left[C_1^2(1+\lambda)^2-1\right] - 8C_1(C_1-1)^2(\lambda C_1)(1+\lambda) + \left[(C_1-1)^2 - 2C_1(1+\lambda)(\lambda C_1)\right]^2\right\}$$

$$+\frac{1}{4}\left\{(C_1-1)^3(\lambda C_1) - 3C_1(C_1-1)(1+\lambda)(\lambda C_1)^2\right\} + \frac{1}{9}\left\{3(C_1-1)^2(\lambda C_1)^2 - 2C_1(1+\lambda)(\lambda C_1)^3\right\}$$

$$+\frac{1}{5}(C_1-1)(\lambda C_1)^3 + \frac{1}{22}(\lambda C_1)^4$$

multiplied by β^6

$$\int_0^\beta \left[\left\{\frac{d\Theta}{d\alpha}-1\right\}^2 + \left\{\frac{\Theta}{\alpha}-1\right\}^2\right]\alpha\,d\alpha = \left[C_1(1+\lambda)-1\right]^2 - 2(C_1-1)\left[C_1(1+\lambda)-1\right]$$

$$+\frac{5}{3}(C_1-1)^2 - 2(\lambda C_1)\left[C_1(1+\lambda)-1\right] + 4(C_1-1)(\lambda C_1) + \frac{13}{5}(\lambda C_1)^2$$

multiplied by β^2

$$\int_0^\beta \alpha^2\Theta\,d\alpha = \frac{C_1}{4}(1+\lambda) - \frac{1}{6}(C_1-1) - \frac{1}{8}(\lambda C_1)$$

multiplied by β^4

2(4)

$$\frac{2}{3}(1+\lambda)^2\left[(1+\lambda)^2 C_1^3 - C_1\right] - \frac{1}{2}(1+\lambda)\left[4(1+\lambda)^2 C_1^3 - 3(1+\lambda)^2 C_1^2 - 2C_1 + 1\right]$$

$$+\frac{2}{5}\left\{2\left[3(1+\lambda)^2 - 2\lambda(1+\lambda)^3\right]C_1^3 - 3\left[3(1+\lambda)^2 - (1+\lambda)^3\right]C_1^2 + \left[3(1+\lambda)^2 + 2\lambda(1+\lambda) - 1\right]C_1 + \left[1-(1+\lambda)\right]\right\}$$

$$+\frac{1}{3}\left\{4\left[\overset{\lambda}{3}(1+\lambda)^2 - (1+\lambda)\right]C_1^3 - 9\left[(1+\lambda)^3 - (1+\lambda)\right]C_1^2 + 2\left[3(1+\lambda)^2 - 3(1+\lambda) - \lambda\right]C_1 + 2(1+\lambda)\right\}$$

$$+\frac{2}{7}\left\{\left[6\lambda^2(1+\lambda)^2 - 12\lambda(1+\lambda) + 1\right]C_1^3 - \left[3\lambda(1+\lambda)^2 - 6(1+2\lambda)(1+\lambda) - 3\lambda + 6\lambda^2(1+\lambda)\right]C_1^2\right.$$

$$\left. + \left[\lambda(1+2\lambda) - 4(1+\lambda)(2+\lambda) + (1-2\lambda)\right]C_1 + \left[2\lambda + 2(1+\lambda)\right]\right\}$$

$$+\frac{1}{4}\left\{4\lambda\left[1-3\lambda(1+\lambda)\right]C_1^3 - 3\left[(3\lambda+1) - 3\lambda(1+\lambda)(2+\lambda)\right]C_1^2 + 2(1+\lambda)\left[3-(2\lambda+1)\right]C_1 + 2\lambda\right\}$$

$$+\frac{2}{9}\left\{2\lambda^2\left[3-2\lambda(1+\lambda)\right]C_1^3 - 9\lambda(1+\lambda)(1-\lambda)C_1^2 + 3\left[1+2\lambda-\lambda^2\right]C_1 - 2(1+\lambda)\right\}$$

$$+\frac{1}{5}\left\{4\lambda^3 C_1^3 - 3\lambda^2(3+\lambda)C_1^2 + 6\lambda(1+\lambda)C_1 - (1+3\lambda)\right\}$$

$$+\frac{2}{11}\lambda\left\{\lambda^3 C_1^3 - 3\lambda^2 C_1^2 + 3\lambda C_1 \oplus 1\right\} = I_1$$

$$2(1+\lambda)^2 C_1 - 2\left[2(1+\lambda)C_1 - (2+\lambda)\right] + \frac{10}{3}(C_1 - 1) - 2\left[2\lambda(1+\lambda)C_1 - (1+2\lambda)\right]$$

$$+ 4\left[2\lambda C_1 - (1+\lambda)\right] + \frac{26}{5}\lambda(\lambda C_1 - 1) = I_2$$

$$\frac{1+\lambda}{4} - \frac{1}{6} - \frac{\lambda}{8} = \frac{1}{12} + \frac{\lambda}{8} \quad I_3$$

$$\beta^2 \frac{E\left(\frac{t}{r}\right)}{4}\left\{\overbrace{A_3C_1^3+A_2C_1^2+A_1C_1+A_0}^{I_1}\right\}+\frac{E\left(\frac{t}{r}\right)^3}{12}\frac{1}{\beta^2}\left\{\overbrace{B_1C_1+B_0}^{I_2}\right\}+\frac{p}{4}\left\{\underbrace{\frac{1}{3}+\frac{\lambda}{2}}_{I_3}\right\}=0. \quad (215)$$

$$\beta^2\left\{A_3C_1^3+A_2C_1^2+A_1C_1+A_0\right\}+\frac{\left(\frac{t}{r}\right)^2}{3}\frac{1}{\beta^2}\left\{B_1C_1+B_0\right\}+\left(\frac{2\sigma}{E}\right)\left(\frac{1}{3}+\frac{\lambda}{2}\right)=0.$$

$$\beta^2=\frac{\left(\frac{t}{r}\right)}{\sqrt{3}}\left(B_1C_1+B_0\right)^{\frac{1}{2}}\left(A_3C_1^3+A_2C_1^2+A_1C_1+A_0\right)^{-\frac{1}{2}}$$

$$\therefore \quad \frac{1}{\sqrt{3}}\left[\left(B_1C_1+B_0\right)\left(A_3C_1^3+A_2C_1^2+A_1C_1+A_0\right)\right]^{\frac{1}{2}}+\phi\left(\frac{1}{3}+\frac{\lambda}{2}\right)=0.$$

$$B_1\left(A_3C_1^3+A_2C_1^2+A_1C_1+A_0\right)+\left(B_1C_1+B_0\right)\left(3A_3C_1^2+2A_2C_1+A_1\right)=0$$

or $\quad (4A_3B_1)C_1^3+3(B_1A_2+B_0A_3)C_1^2+2(A_1B_1+A_2B_0)C_1+(A_1B_0+B_1A_0)=0$

$$A_3=\frac{2}{3}(1+\lambda)^4-2(1+\lambda)^3+\frac{4}{5}(1+\lambda)^2(3-2\lambda-2\lambda^2)+\frac{4}{3}(1+\lambda)\left(\cancel{2+3\lambda}\right)$$

$\frac{4}{3}(1+\lambda)(3\lambda+3\lambda^2-1)$

$$+\frac{2}{7}\left[6\lambda^2(1+\lambda)^2-12\lambda(1+\lambda)+1\right]+\lambda(1-3\lambda-3\lambda^2)+\frac{4\lambda^2}{9}\left[3-2\lambda-4\lambda^2\right]$$

$$+\frac{4}{5}\lambda^3+\frac{2\lambda^4}{11}$$ OK ✓✓

$$A_2= \text{——} \left\{-\frac{3}{2}(1+\lambda)^3+\frac{6}{5}(1+\lambda)^2(2-\lambda)+3\lambda(1+\lambda)(2+\lambda)+\frac{6}{7}(3\lambda^3-6\lambda-2)\right.$$

(annotation: "should be ⊖" pointing to 6λ; OK)

$$\left.\overset{\text{OK}}{\textcircled{+}}\frac{3}{4}\left[(3\lambda+1)-3\lambda(1+\lambda)(2+\lambda)\right]\overset{\text{OK}}{\textcircled{+}}2\lambda(1+\lambda)(1-\lambda)+\frac{3}{5}\lambda^2(3+\lambda)+\frac{6}{11}\lambda^3\right\}$$ OK

$$A_1=\left\{-\frac{2}{3}(1+\lambda)^2+(1+\lambda)+\frac{2}{5}(5\lambda^2+8\lambda+2)+\frac{2}{3}\lambda(3\lambda+2)+\frac{2}{7}\cancel{(7+13\lambda)}\right.$$

$-\frac{2}{7}(2\lambda^2+24\lambda+15)$ OK

I think this should be $-\frac{4}{7}(2\lambda^2+6\lambda+3)$

$$\left.+(1+\lambda)(1-\lambda)+\frac{2}{3}(1+2\lambda-\lambda^2)+\frac{6}{5}\lambda(1+\lambda)+\frac{6\lambda^2}{11}\right\}$$

I think this should be $-3\lambda(1+\lambda)$ (O.K.)

2(b)

OK OK

$$A_0 = -\frac{1}{2}(1+\lambda) - \frac{2}{5}\lambda + \frac{2}{3}(1+\lambda) + \frac{4}{7}(1+2\lambda) + \frac{1}{2}\left(-\frac{4}{9}(1+\lambda)\right) - \frac{1}{5}(1+3\lambda)$$

$$+\frac{2}{11}\lambda$$

↖ should be $\ominus$

$$B_1 = 2(1+\lambda)^2 - 4(1+\lambda) + \overset{OK}{\frac{10}{3}} - 4\lambda(1+\lambda) + \overset{OK}{8\lambda} + \overset{OK}{\frac{26}{5}}\lambda^2$$ OK

$$B_0 = 2(2+\lambda) - \overset{OK}{\frac{10}{3}} + 2(1+2\lambda) - 4(1+\lambda) - \overset{OK}{\frac{26}{5}}\lambda$$ OK

A_3

①	②	③	④	⑤	⑥	⑦	⑧	⑨
λ	$1+\lambda$	$(1+\lambda)^2$	$(1+\lambda)^3$	$(1+\lambda)^4$	$\frac{2}{3}$ ⑤	-2 ④	$\lambda(1+\lambda)$	2 ⑧
-0.5	0.5	0.25	0.125	0.0625	0.04167	-0.250	-0.250	-0.500
-0.4	0.6	0.36	0.216	0.1296	0.08640	-0.432	-0.240	-0.480
-0.3	0.7	0.49	0.343	0.2401	0.16007	-0.686	-0.210	-0.420
-0.2	0.8	0.64	0.512	0.4096	0.27307	-1.024	-0.160	-0.320
-0.1	0.9	0.81	0.729	0.6561	0.43740	-1.458	-0.090	-0.180
0	1.0	1.00	1.000	1.0000	0.66667	-2.000	0	0
+0.1	1.1	1.21	1.331	1.4641	0.97607	-2.662	+0.110	+0.220
+0.2	1.2	1.44	1.728	2.0736	1.38240	-3.456	+0.240	+0.480
+0.3	1.3	1.69	2.197	2.8561	1.90407	-4.394	+0.390	+0.780
+0.4	1.4	1.96	2.744	3.8416	2.56107	-5.488	+0.560	+1.120
+0.5	1.5	2.25	3.375	5.0625	3.36834	-6.750	+0.750	+1.500

	⑩	⑪	⑫	⑬	⑭	⑮	⑯	⑰
λ	$3-$ ⑨	$\frac{4}{5}$ ⑩	⑪ × ③	$3\times$ ⑧	⑬ -1	$\frac{4}{3}\times$ ⑭	⑮ × ②	λ^2
-0.5	3.500	2.800	0.7000	-0.750	-1.750	-2.33333	-1.16667	0.25
-0.4	3.480	2.784	1.00224	-0.720	-1.720	-2.29333	-1.37599	0.16
-0.3	3.420	2.736	1.34064	-0.630	-1.630	-2.17333	-1.52133	0.09
-0.2	3.320	2.656	1.69984	-0.480	-1.480	-1.97333	-1.57866	0.04
-0.1	3.180	2.544	2.06064	-0.270	-1.270	-1.69333	-1.52399	0.01
0	3.000	2.400	2.40000	0	-1.000	-1.33333	-1.33333	0
0.1	2.780	2.224	2.69104	+0.330	-0.670	-0.89333	-0.98266	0.01
0.2	2.520	2.016	2.90304	+0.720	-0.280	-0.37333	-0.44799	0.04
0.3	2.220	1.776	3.00144	+1.170	+0.170	+0.22667	+0.29467	0.09
0.4	1.880	1.504	2.94784	+1.680	+0.680	+0.90667	+1.26934	0.16
0.5	1.500	1.200	2.70000	+2.250	+1.250	+1.66667	+2.50000	0.25

A_4

	(18)	(19)	(20)	(21)	(22)	(23)	(24)
λ	(8) - 2	2 × (13)	(18) × (19)	(20) + 1	$\frac{2}{7}$ (21)	$-\lambda \cdot$ (14)	$4\lambda^2$
-0.5	-2.250	-1.500	+3.3750	4.3750	1.25000	-0.875	1.000
-0.4	-2.240	-1.440	+3.2256	4.2256	1.20731	-0.688	0.640
-0.3	-2.210	-1.260	+2.7846	3.7846	1.08131	-0.489	0.360
-0.2	-2.160	-0.960	+2.0736	3.0736	0.87817	-0.296	0.160
-0.1	-2.090	-0.540	+1.1286	2.1286	0.60817	-0.127	0.04
0	-2.000	0	0	1.0000	0.28571	0	0
+0.1	-1.890	+0.660	-1.2474	-0.2474	-0.07069	+0.067	0.04
+0.2	-1.760	+1.440	-2.5344	-1.5344	-0.43840	+0.056	0.16
+0.3	-1.610	+2.340	-3.7674	-2.7674	-0.79068	-0.051	0.36
+0.4	-1.440	+3.360	-4.8384	-3.8384	-1.09668	-0.272	0.64
+0.5	-1.250	+4.500	-5.6250	-4.6250	-1.32143	-0.625	1.00

	(25)	(26)	(27)	(28)	(29)	(30)	(31)
λ	2λ	(25) + (24)	3 - (26)	$\frac{4}{9}$ × (27)	(17) × 28	$\frac{4}{5}\lambda^3$	λ^4
-0.5	-1.0	0	3.000	1.33333	0.33333	-0.1000	0.0625
-0.4	-0.8	-0.16	3.160	1.40444	0.22471	-0.0512	0.0256
-0.3	-0.6	-0.24	3.240	1.44000	0.1296	-0.0216	0.0081
-0.2	-0.4	-0.24	3.240	1.44000	0.05760	-0.0064	0.0016
-0.1	-0.2	-0.16	3.160	1.40444	0.01404	-0.0008	0.0001
0	0	0	3.000	1.33333	0	0	0
+0.1	+0.2	+0.24	2.760	1.22666	0.012267	+0.0008	0.0001
+0.2	+0.4	+0.56	2.440	1.08444	0.043378	+0.0064	0.0016
+0.3	+0.6	+0.96	2.040	0.90667	0.081600	+0.0216	0.0081
+0.4	+0.8	+1.44	1.560	0.69333	0.110933	+0.0512	0.0256
+0.5	+1.0	+2.00	1.000	0.44444	0.111111	+0.1000	0.0625

A_3

	(32)	(33)				
λ	$\frac{2}{\pi}\lambda^4$	(A_3)				
−0.5	0.01136	−0.05531				
−0.4	0.00465	−0.02188				
−0.3	0.00147	−0.00484				
−0.2	0.00029	+0.00391				
−0.1	0.00002	+0.01048				
0	0	+0.01905				
+0.1	0.00002	+0.03185				
+0.2	0.00029	+0.04912				
+0.3	0.00147	+0.06917				
+0.4	0.00465	+0.08835				
+0.5	0.01136	+0.09438				

$$A_2 = -\Big\{ -1.5(\lambda^3+3\lambda^2+3\lambda+1) + 1.2(-\lambda^3+3\lambda+2) + 3(\lambda^3+3\lambda^2+2\lambda) \qquad (220)$$

$$+\ 0.85714286(3\lambda^3-6\lambda-2) + 0.75(-3\lambda^3-9\lambda^2-3\lambda+1)$$

$$+\ 2(-\lambda^3+\lambda) + 0.6(\lambda^3+3\lambda) + 0.54545454\,\lambda^3 \Big\}$$

$$= \Big\{ (1.5+1.2-3-2.5714286+2.25+2-0.6-0.545454)\lambda^3$$

$$+\ (4.5 \quad -9+6.75)\lambda^2$$

$$+\ (4.5-3.6-6+5.1428571+2.25-2-1.8)\lambda$$

$$+\ (1.5-2.4+1.7142857-0.75) \Big\}$$

$$\boxed{A_2 = 0.23312\,\lambda^3 + 2.25000\,\lambda^2 - 1.50714\,\lambda + 0.06429}$$

$$A_1 = \Big\{ -0.666667(\lambda^2+2\lambda+1) + (\lambda+1) + 0.40(5\lambda^2+8\lambda+2) + 0.666667(3\lambda^2+2\lambda)$$

$$-\ \frac{2}{7}(7\lambda^2+24\lambda+15) - 3(\lambda^2+\lambda) + \frac{2}{3}(-\lambda^2+2\lambda+1) + 1.2(\lambda^2+\lambda)$$

$$+\ 0.54545454\,\lambda^2 \Big\}$$

$$= \Big\{ (0.666667 + 2 - 2 - 3 + 1.2 + 0.545454)\lambda^2$$

$$+\ (+1.33333 + 1 + 3.2 - 6.85714 - 3 + 1.2)\lambda$$

$$+\ (1 + 0.8 - 4.285714) \Big\}$$

$$\boxed{A_1 = -0.58788\,\lambda^2 - 3.12381\,\lambda - 2.48571}$$

221)

$$A_0 = (-0.5 - 0.4 + 0.666667 + 1.142857 + 0.5 - 0.4444444 - 0.6 + 0.1111111)\lambda$$

$$(-0.5 + 0.666667 + 0.5714286 - 0.444444 - 0.2)$$

$$\boxed{A_0 = 0.25398\lambda + 0.09366}$$

$$B_1 = (2 - 4 + 5.2)\lambda^2 + (4 - 4 - 4 + 8)\lambda + (2 - 4 + 3.3333)$$

$$\boxed{B_1 = 3.2\lambda^2 + 4\lambda + 1.33333}$$

$$B_0 = (2 + 4 - 4 - 5.2)\lambda + (4 - 3.3333 + 2 - 4)$$

$$\boxed{B_0 = -3.2\lambda - 1.33333}$$

①	②	③	④	⑤	⑥	⑦
λ	λ^2	λ^3	$0.23312\lambda^3$	$2.25000\lambda^2$	-1.50714λ	A_2
-0.5	0.25	-0.125	-0.02914	0.56250	+0.75357	1.35122
-0.4	0.16	-0.064	-0.01492	0.36000	+0.60286	1.01223
-0.3	0.09	-0.027	-0.00629	0.20250	+0.45214	0.71264
-0.2	0.04	-0.008	-0.00186	0.09000	+0.30143	0.45386
-0.1	0.01	-0.001	-0.00023	0.02250	+0.15071	0.23727
0	0	0	0	0	0	0.06429
+0.1	0.01	+0.001	+0.00023	0.02250	-0.15071	-0.06369
+0.2	0.04	+0.008	+0.00186	0.09000	-0.30143	-0.14528
+0.3	0.09	+0.027	+0.00629	0.20250	-0.45214	-0.17906
+0.4	0.16	+0.064	+0.01492	0.36000	-0.60286	-0.16365
+0.5	0.25	+0.125	+0.02914	0.56250	-0.75357	-0.09764

	⑧	⑨	⑩	⑪	⑫
λ	$-0.58788\lambda^2$	-3.12381λ	A_1	0.25398λ	A_0
-0.5	-0.14697	+1.56191	-1.07077	-0.12699	-0.03333
-0.4	-0.09406	+1.24952	-1.33025	-0.10159	-0.00793
-0.3	-0.05291	+0.93714	-1.60148	-0.07619	+0.01747
-0.2	-0.02352	+0.62476	-1.88447	-0.05080	+0.04286
-0.1	-0.00588	+0.31238	-2.17921	-0.02540	+0.06826
0	0	0	-2.48571	0	+0.09366
+0.1	-0.00588	-0.31238	-2.80397	+0.02540	+0.11906
+0.2	-0.02352	-0.62476	-3.13399	+0.05080	+0.14446
+0.3	-0.05291	-0.93714	-3.47576	+0.07619	+0.16985
+0.4	-0.09406	-1.24952	-3.82929	+0.10159	+0.19525
+0.5	-0.14697	-1.56191	-4.19459	+0.12699	+0.22065

223)

K_1

①	②	③	④	⑤	⑥	⑦
λ	$3.2\lambda^2$	4λ	B_1	B_0	A_3B_1	$4A_3B_1$
-0.5	0.8	-2.0	+0.13333	+0.26667	-0.007374	-0.02950
-0.4	0.512	-1.6	+0.24533	-0.05333	-0.005368	-0.02147
-0.3	0.288	-1.2	+0.42133	-0.37333	-0.002039	-0.00816
-0.2	0.128	-0.8	+0.66133	-0.69333	+0.002586	+0.01034
-0.1	0.032	-0.4	+0.96533	-1.01333	+0.010117	+0.04047
0	0	0	+1.33333	-1.33333	+0.025400	+0.10160
+0.1	0.032	+0.4	+1.76533	-1.65333	+0.056226	+0.22490
+0.2	0.128	+0.8	+2.26133	-1.97333	+0.111077	+0.44431
+0.3	0.288	+1.2	+2.82133	-2.29333	+0.195151	+0.78060
+0.4	0.512	+1.6	+3.44533	-2.61333	+0.304395	+1.21758
+0.5	0.800	+2.0	+4.13333	-2.93333	+0.390104	+1.56042

	⑧	⑨	⑩	⑪	⑫	⑬
λ	A_2B_1	A_3B_0	⑧+⑨	3⑩ K_2	A_1B_1	A_2B_0
-0.5	+0.18016	-0.01475	+0.16541	+0.49623	-0.14277	+0.36033
-0.4	+0.24833	+0.00117	+0.24950	+0.74850	-0.32635	-0.05398
-0.3	+0.30026	+0.00181	+0.30207	+0.90621	-0.67475	-0.26605
-0.2	+0.30015	-0.00271	+0.29744	+0.89232	-1.24626	-0.31467
-0.1	+0.22904	-0.01062	+0.21842	+0.65526	-2.10366	-0.24043
0	+0.08572	-0.02540	+0.06032	+0.18096	-3.31427	-0.08572
+0.1	-0.11243	-0.05266	-0.16509	-0.49527	-4.94993	+0.10530
+0.2	-0.32853	-0.09693	-0.42546	-1.27638	-7.08699	+0.28669
+0.3	-0.50519	-0.15863	-0.66382	-1.99146	-9.80627	+0.41064
+0.4	-0.56383	-0.23089	-0.79472	-2.38416	-13.19317	+0.42767
+0.5	-0.40358	-0.27685	-0.68043	-2.04129	-17.33762	+0.28641

	K_3			K_4		
	(14)	(15)	(16)	(17)	(18)	
λ	(12)+(13)	2×(14)	$A_0 B_1$	$A_1 B_0$	(16)+(17)	
−0.5	+0.21756	+0.4351	−0.004444	−0.285542	−0.28998	
−0.4	−0.27237	−0.5448	−0.001945	+0.070942	0.06899	
−0.3	−0.94080	−1.8816	+0.006097	+0.597881	0.60398	
−0.2	−1.56093	−3.1218	+0.028345	+1.30656	1.3349	
−0.1	−2.34409	−4.6882	+0.065893	+2.20826	2.2742	
0	−3.39999	−6.7999	+0.124880	+3.31427	3.4392	
+0.1	−4.84463	−9.6893	+0.210180	+4.63589	4.8461	
+0.2	−6.80030	−13.6006	+0.326672	+6.18440	6.5111	
+0.3	−9.39563	−18.7913	+0.479203	+7.97106	8.4503	
+0.4	−12.76550	−25.5310	+0.672701	+10.0072	10.6799	
+0.5	−17.05121	−34.1024	+0.912019	+12.3041	13.2161	

	(19)	(20)	(21)	(22)	(23)	(24)
λ	α_1	α_2	α_3	$\xi = -\frac{\alpha_1}{3}$	(11)×(22)	p=(20)+(23)
−0.5	−16.821	−14.749	+9.8298	+5.6070	−94.315	−109.064
−0.4	−34.863	+25.375	−3.2133	+11.621	−405.143	−379.768
−0.3	−111.06	+230.59	−74.017	+37.020	−4111.44	−3880.85
−0.2	+86.298	−301.91	+129.10	−28.766	−2482.45	−2784.36
−0.1	+16.191	−115.84	+56.195	−5.3970	−87.383	−203.223
0	+1.7811	−66.928	+33.850	−0.59370	−1.0574	−67.985
+0.1	−2.2022	−43.083	+21.548	+0.73407	−1.6166	−44.700
+0.2	−2.8727	−30.611	+14.654	+0.95757	−2.7508	−33.362
+0.3	−2.5512	−24.073	+10.825	+0.85040	−2.1695	−26.243
+0.4	−1.9581	−20.969	+8.7714	+0.65270	−1.2781	−22.247
+0.5	−1.3082	−21.855	+8.4696	+0.43607	−0.57047	−22.425

$$C_1^3 + \alpha_1 C_1^2 + \alpha_2 C_1 + \alpha_3 = 0$$

Put $\quad C_1 = x + \xi$

$$x^3 + 3x^2\xi + 3x\xi^2 + \xi^3 + \alpha_1(x^2 + 2x\xi + \xi^2) + \alpha_2(x+\xi) + \alpha_3 = 0$$

$$x^3 + \underbrace{\{3\xi + \alpha_1\}}_{=0} x^2 + \underbrace{\{3\xi^2 + 2\alpha_1\xi + \alpha_2\}}_{p} x + \underbrace{\{\xi^3 + \alpha_1\xi^2 + \alpha_2\xi + \alpha_3\}}_{q} = 0.$$

$$\xi = -\frac{\alpha_1}{3}$$

$$p = 3\xi^2 + 2\alpha_1\xi + \alpha_2 = \frac{\alpha_1^2}{3} - \frac{2}{3}\alpha_1^2 + \alpha_2 = \alpha_2 - \frac{\alpha_1^2}{3}$$

$$q = \xi^3 + \alpha_1\xi^2 + \alpha_2\xi + \alpha_3 = -\frac{\alpha_1^3}{27} + \frac{\alpha_1^3}{9} - \frac{\alpha_1\alpha_2}{3} + \alpha_3$$

$$\boxed{\begin{aligned} p &= \alpha_2 - \frac{\alpha_1^2}{3} \\ q &= \frac{2}{27}\alpha_1^3 - \frac{\alpha_1\alpha_2}{3} + \alpha_3 \end{aligned}}$$

$$q = \frac{\alpha_1}{3}\left[\frac{2}{9}\alpha_1^2 - \alpha_2\right] + \alpha_3$$

22b)

λ	(25) $-\alpha_1$ (23)	(26) $\frac{2}{27}\alpha_1^3$	(27) (22)×(20)	(28) q = (26)+(27)+(21)	(29) $-p/3$	(30) $r^{\frac{1}{3}}$ = (29)$^{\frac{1}{2}}$
−0.5	−1586.47	−352.549	−82.698	−425.417	36.355	6.0295
−0.4	−14124.5	−3138.78	+294.883	−2847.11	126.589	11.251
−0.3	−456616.5	−101470.3	+8536.44	−93007.9	1293.62	35.967
−0.2	+214230.5	+47606.77	+8684.74	+56420.6	928.120	30.465
−0.1	+1414.82	+314.404	+625.188	+995.787	67.7410	8.2305
0	+1.88334	+0.41852	+39.7352	+74.004	22.662	4.7605
0.1	−3.56008	−0.79113	−31.6259	−10.869	14.900	3.8601
0.2	−7.90222	−1.75605	−29.3122	−16.414	11.121	3.3348
0.3	−5.53483	−1.22996	−20.4717	−10.877	8.7477	2.9576
0.4	−2.50265	−0.55614	−13.6865	−5.472	7.4157	2.7232
0.5	−0.74629	−0.16584	−9.5303	−1.2265	7.4750	2.7341

λ	(31) r	(32) $-q/2$ ÷ (31)	(33) θ	(34) β_1	(35) β_2	(36) β_3
−0.5	219.20	+0.97039	13°58.7′	4°39.6′	124°39.6′	244°39.6′
−0.4	1424.3	+0.99948	1°51.0′	37′	120°37′	240°37′
−0.3	46527	+0.99951	1°48.0′	36′	120°36′	240°36′
−0.2	28275	−0.99771	183°52.5′	61°17.5′	181°17.5′	301°17.5′
−0.1	557.54	−0.89302	206°44.7′	68°52.9′	188°52.9′	308°52.9′
0	107.88	−0.31518	102°22.3′	34°7.4′	154°7.4′	274°7.4′
+0.1	57.515	+0.094488	84°34.7′	28°11.6′	148°11.6′	268°11.6′
+0.2	37.086	+0.22130	77°12.9′	25°44.3′	145°44.3′	265°44.3′
+0.3	25.873	+0.21020	77°52.0′	25°57.3′	145°57.3′	265°57.3′
+0.4	20.194	+0.13549	82°12.8′	27°24.3′	147°24.3′	267°24.3′
+0.5	20.437	+0.030007	88°16.8′	29°25.6′	149°25.6′	269°25.6′

λ	(37) $\cos\beta_1$	(38) $\cos\beta_2$	(39) $\cos\beta_3$	(40) x_1	(41) x_2	(42) x_3
−0.5	0.99669	−0.56870	−0.42799	−6.8580	−5.1611	+12.019
−0.4	0.99994	−0.50929	−0.49065	−11.460	−11.041	+22.501
−0.3	0.99995	−0.50904	−0.49090	−36.617	−35.312	+71.930
−0.2	0.48035	−0.99975	+0.51940	−60.915	+29.268	+31.647
−0.1	0.36030	−0.98801	+0.62772	−16.264	+5.9309	+10.3329
0	0.82783	−0.90100	+0.07481	−8.5784	+0.71227	+7.8818
0.1	0.88136	−0.84983	−0.03153	−6.5609	−0.24342	+6.8043
0.2	0.90078	−0.82648	−0.07431	−5.5123	−0.49562	+6.0078
0.3	0.89914	−0.82844	−0.07054	−4.9004	−0.41726	+5.3186
0.4	0.88778	−0.84250	−0.04527	−4.5886	−0.24656	+4.8352
0.5	0.87099	−0.86098	−0.01011	−4.7080	−0.05528	+4.7627

λ	(43) $C_1^{(1)}$	(44) $C_1^{(2)}$	(45) $C_1^{(3)}$
−0.5	−1.2510	+0.4459	+17.626
−0.4	+0.161	+0.580	+34.122
−0.3	+0.403	+1.708	+108.95
−0.2	−89.681	+0.502	+2.881
−0.1	−21.661	+0.5339	+4.9359
0	−9.1721	+0.11857	+7.2881
0.1	−5.8268	+0.49065	+7.5384
0.2	−4.5547	+0.46195	+6.9654
0.3	−4.0500	+0.43314	+6.1690
0.4	−3.9359	+0.40614	+5.4879
0.5	−4.2719	+0.38079	+5.1988

	①	②	③	④	⑤
λ	$[C_1^{(1)}]^2$	$[C_1^{(2)}]^2$	$[C_1^{(3)}]^2$	$[C_1^{(1)}]^3$	$[C_1^{(2)}]^3$
−0.5	1.5650	+0.19883	+310.68	−1.9578	+0.088658
−0.4	+0.02592	+0.3364	+1164.31	+0.004173	+0.19511
−0.3	+0.16241	+2.9173	+11870.1	+0.065451	+4.9827
−0.2	+8042.7	+0.2520	+8.3002	−721276	+0.12650
−0.1	+469.20	+0.28505	+24.3631	−10163.3	+0.152188
0	+84.1274	+0.014059	+53.116	−771.625	+0.0016670
+0.1	+33.951	+0.24069	+56.827	−197.829	+0.11809
+0.2	+20.745	+0.21340	+48.517	−94.487	+0.09858
+0.3	+16.403	+0.18761	+38.057	−66.432	+0.08126
0.4	+15.491	+0.16495	+30.117	−60.971	+0.06699
0.5	+18.249	+0.14500	+27.028	−77.958	+0.05521

	⑥
λ	$[C_1^{(3)}]^3$
−0.5	5475.98
−0.4	39728.6
−0.3	1293247
−0.2	+23.913
−0.1	+120.254
0	+387.12
+0.1	+428.38
0.2	+337.94
0.3	+234.77
0.4	+165.28
0.5	+140.51

⑤ = B_0

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λ	① $B_1C_1^{(1)}$	② ⑤+①	③ $B_1C_1^{(2)}$	④ ⑤+③	⑤ $B_1C_1^{(3)}$	⑥ ⑤+⑤
−0.5	−0.16680	+0.09987	+0.05945	+0.32612	+2.3500	2.6167
−0.4	+0.03950	−0.01383	+0.14229	+0.08896	+8.3712	8.3179
−0.3	+0.16980	−0.20353	+0.71963	+0.34630	+45.904	45.531
−0.2	−59.309	−60.002	+0.33199	−0.36134	+1.9053	1.2120
−0.1	−20.910	−21.923	+0.51539	−0.49794	+4.7648	3.7515
−0.[illegible]	−12.229	−13.562	+0.15809	−1.17524	+9.7174	8.3841
+0.1	−10.286	−11.939	+0.86616	−0.78717	+13.3077	11.6544
+0.2	−10.300	−12.273	+1.04462	−0.92871	+15.7511	13.7778
+0.3	−11.426	−13.719	+1.2220	−1.0711	+17.405	15.112
+0.4	−13.560	−16.173	+1.3993	−1.2140	+18.908	16.295
+0.5	−17.657	−20.590	+1.5739	−1.3594	+21.488	18.555

$$(B_1C_1+B_0)(A_3C_1^3+A_2C_1^2+A_1C_1+A_0)$$

$$=-\frac{(B_1C_1+B_0)^2(3A_3C_1^2+2A_2C_1+A_1)}{B_1}$$

$$\phi=\frac{1}{\left(\frac{1}{3}+\frac{1}{2}\right)}\left\{-(B_1C_1+B_0)\right\}\left\{\frac{-(3A_3C_1^2+2A_2C_1+A_1)}{3B_1}\right\}^{\frac{1}{2}}$$

1

λ	① $3A_3[C_1^{(1)}]^2$	② $2A_2\ C_1^{(1)}$	③ A_1 + ① + ②	④ $-$③$/3B_1$
−0.5	− 0.25968	− 33.808	− 4.7113	11.778
−0.4	− 0.001701	+ 0.3259	− 1.0061	1.3670
−0.3	− 0.002358	+ 0.57439	− 1.0295	0.81448
−0.2	+ 94.341	− 81.405	+ 11.052	
−0.1	+ 14.752	− 10.279	+ 2.294	
0	+ 4.8079	− 1.1793	+ 1.1429	
0.1	+ 3.2440	+ 0.74221	+ 1.1822	
0.2	+ 3.0570	+ 1.3234	+ 1.2464	
0.3	+ 3.4038	+ 1.4504	+ 1.3784	
0.4	+ 4.1062	+ 1.2882	+ 1.5651	
0.5	+ 5.1670	+ 0.8342	+ 1.8066	

λ	⑤ $3A_3[C_1^{(2)}]^2$	⑥ $2A_2\ C_1^{(2)}$	⑦ A_1 + ⑤ + ⑥	⑧ $-$⑦$/3B_1$
−0.5	− 0.03299	+ 1.2050	+ 0.1012	
−0.4	− 0.02208	+ 1.1742	− 0.1782	0.2421
−0.3	− 0.042359	+ 2.4344	+ 0.7905	
−0.2	+ 0.002955	+ 0.4557	− 1.4258	0.7186
−0.1	+ 0.008962	+ 0.25336	− 1.9169	0.6619
0	+ 0.0008035	+ 0.01525	− 2.4697	0.6174
0.1	+ 0.022998	− 0.06250	− 2.8435	0.5369
0.2	+ 0.031447	− 0.13422	− 3.2368	0.4771
0.3	+ 0.038931	− 0.15512	− 3.5920	0.4244
0.4	+ 0.043720	− 0.13293	− 3.9185	0.3791
0.5	+ 0.041055	− 0.07436	− 4.2279	0.3410

2

3

λ	⑨ $3A_3[C_1^{(3)}]^2$	⑩ $2A_2 C_1^{(3)}$	⑪ A_1 + ⑨ + ⑩	⑫ − ⑪ / $3B_1$
−0.5	− 51.551	+ 47.633	−4.989	12.473
−0.4	− 76.425	+ 69.079	−8.676	11.788
−0.3	− 172.35	+270.09		
−0.2	+ 0.09736	+ 2.6151		
−0.1	+ 0.76598	+ 2.3423		
0	+ 3.0356	+0.93710	+	
0.1	+ 5.4298	− 0.96024		
0.2	+ 7.1495	−7.0239		
0.3	+ 7.8972	− 2.2092		
0.4	+ 7.9825	−1.7962		
0.5	+ 7.6527	− 1.0152		

λ	⑬ $6/2+3\lambda$	⑭ $\phi^{(1)}$	⑮ $\phi^{(2)}$	⑯ $\phi^{(3)}$
−0.5	12	− 4.11		
−0.4	7.5	+ 0.1213		
−0.3	5.45454	+1.000		
−0.2	4.28571			
−0.1	3.5294			
0	3			
0.1	2.60870			
0.2				
0.3				
0.4				
0.5				

$$\frac{2}{3}(1+\lambda)^2\left[(1+\lambda)^2 c_1^3 - c_1\right] - \frac{1}{2}(1+\lambda)\left[4(1+\lambda)^2 c_1^3 - 3(1+\lambda)^2 c_1^2 - 2c_1 + 1\right] \qquad 233)$$

$$+\frac{2}{5}\left\{2(1+\lambda)^2\left[3-2\lambda(1+\lambda)\right]c_1^3 - 9(1+\lambda)^2 c_1^2 + \left[3(1+\lambda)^2+\lambda(1+\lambda)-1\right]c_1 + 1\right\}$$

$$+\frac{1}{3}\left\{4(1+\lambda)\left[3\lambda(1+\lambda)-1\right]c_1^3 + 9(1+\lambda)\left[1-\lambda(1+\lambda)\right]c_1^2 - 2\left[\lambda+3(1+\lambda)\right]c_1 + (1+2\lambda)\right\}$$

$$+\frac{2}{7}\left\{\left[6\lambda^2(1+\lambda)^2-12\lambda(1+\lambda)+1\right]c_1^3 + 3\left[6\lambda(1+\lambda)-1\right]c_1^2 - \left[\lambda^2+6\lambda(1+\lambda)-3\right]c_1 - 1\right\}$$

$-4+9-6+3$

$1-3+3-1$

$6-9+2+1$

$$+\frac{1}{4}\left\{4\lambda\left[1-3\lambda(1+\lambda)\right]c_1^3 + 9\lambda\left[\lambda(1+\lambda)-1\right]c_1^2 + 6\lambda c_1 - \lambda\right\}$$

$$+\frac{1}{9}\left\{4\lambda^2\left[3-2\lambda(1+\lambda)\right]c_1^3 - 18\lambda^2 c_1^2 + 6\lambda^2 c_1\right\}$$

$$+\frac{1}{5}\left\{4\lambda^3 c_1^3 - 3\lambda^3 c_1^2\right\} + \frac{2}{11}\lambda^4 c_1^3 = I_1,$$

Multiplied by ρ^6

$$2(1+\lambda)\left[c_1(1+\lambda)-1\right] - 2\left[2(1+\lambda)c_1 - (2+\lambda)\right] + \frac{10}{3}(c_1-1) - 2\lambda\left\{2(1+\lambda)c_1 - 1\right\}$$

$$+ 4\lambda\left[2c_1-1\right] + \frac{26}{5}\lambda^2 c_1 = I_2$$

$$\frac{(1+\lambda)}{4} - \frac{1}{6} - \frac{\lambda}{8} = \frac{1}{12}+\frac{\lambda}{8} = I_3$$

$-4(c_1-1)$

234)

$$A_3 = \frac{2}{3}(1+\lambda)^4 - 2(1+\lambda)^3 + \frac{4}{5}(1+\lambda)^2[3-2\lambda(1+\lambda)] + \frac{4}{3}(1+\lambda)[3\lambda(1+\lambda)-1]$$

$$+ \frac{2}{7}[6\lambda^2(1+\lambda)^2 - 12\lambda(1+\lambda)+1] + \lambda[1-3\lambda(1+\lambda)] + \frac{4}{9}\lambda^2[3-2\lambda(1+\lambda)]$$

$$+ \frac{4}{5}\lambda^3 + \frac{2}{11}\lambda^4$$

$$A_2 = \frac{3}{2}(1+\lambda)^3 - \frac{18}{5}(1+\lambda)^2 + 3(1+\lambda)[1-\lambda(1+\lambda)] + \frac{6}{7}[6\lambda(1+\lambda)-1]$$

$$+ \frac{9}{4}\lambda[\lambda(1+\lambda)-1] - 2\lambda^2 - \frac{3}{5}\lambda^3$$

$$A_1 = -\frac{2}{3}(1+\lambda)^2 + (1+\lambda) + \frac{2}{5}[3(1+\lambda)^2 + \lambda(1+\lambda)-1] - \frac{2}{3}[\lambda + 3(1+\lambda)]$$

$$- \frac{2}{7}[\lambda^2 + 6\lambda(1+\lambda) - 3] + \frac{3}{2}\lambda + \frac{2}{3}\lambda^2$$

$$A_0 = -\frac{1}{2}(1+\lambda) + \frac{2}{5} + \frac{1+2\lambda}{3} - \frac{2}{7} - \frac{\lambda}{4} = -0.083333\,\lambda - 0.052381$$

$$B_1 = 2(1+\lambda)^2 - 4(1+\lambda) + \frac{10}{3} - 4\lambda(1+\lambda) + 8\lambda + \frac{26}{5}\lambda^2$$

$$\boxed{B_0 = \underset{-2(1+\lambda)}{2(2+\lambda)} - \frac{10}{3} - 2\lambda = -2\lambda - 1.33333}$$

$$B_1 = 2(\lambda^2+2\lambda+1) - 4(\lambda+1) + \frac{10}{3} - 4(\lambda^2+\lambda) + 8\lambda + \frac{26}{5}\lambda^2$$

$$= \left(2-4+\frac{26}{5}\right)\lambda^2 + (4-4-4+8)\lambda + \left(2-4+\frac{10}{3}\right)$$

$$\boxed{B_1 = 3.2\,\lambda^2 + 4\lambda + 1.33333}$$

$$A_3 = \frac{2}{3}(\lambda^4+4\lambda^3+6\lambda^2+4\lambda+1) - 2(\lambda^3+3\lambda^2+3\lambda+1) + \frac{4}{5}(3+4\lambda-3\lambda^2-6\lambda^3-2\lambda^4)$$

235)

$$+\frac{4}{3}(3\lambda^3+6\lambda^2+2\lambda-1) + \frac{2}{7}(6\lambda^4+12\lambda^3-6\lambda^2-12\lambda+1) + (\lambda-3\lambda^2-3\lambda^3)$$

$$+\frac{4}{9}(3\lambda^2-2\lambda^3-2\lambda^4) + \frac{4}{5}\lambda^3 + \frac{2}{11}\lambda^4$$

$$A_3 = \left(\frac{2}{3}-\frac{8}{5}+\frac{12}{7}-\frac{8}{9}+\frac{2}{11}\right)\lambda^4 + \left(\frac{8}{3}-2-\frac{24}{5}+4+\frac{24}{7}-3-\frac{8}{9}+\frac{4}{5}\right)\lambda^3$$

$$+\left(4-6-\frac{12}{5}+8-\frac{12}{7}-3+\frac{4}{3}\right)\lambda^2 + \left(\frac{8}{3}-6+\frac{16}{5}+\frac{8}{3}-\frac{24}{7}+1\right)\lambda$$

$$+\left(\frac{2}{3}-2+\frac{12}{5}-\frac{4}{3}+\frac{2}{7}\right)$$

$$A_3 = 0.07389\lambda^4 + 0.20635\lambda^3 + 0.21904\lambda^2 + 0.10476\lambda + 0.019047$$

$$A_2 = \frac{3}{2}(\lambda^3+3\lambda^2+3\lambda+1) - \frac{18}{5}(\lambda^2+2\lambda+1) + 3(-\lambda^3-2\lambda^2+1) + \frac{6}{7}(6\lambda^2+6\lambda-1)$$

$$+\frac{9}{4}(\lambda^3+\lambda^2-\lambda) - 2\lambda^2 - \frac{3}{5}\lambda^3$$

$$A_2 = \left(\frac{3}{2}-3+\frac{9}{4}-\frac{3}{5}\right)\lambda^3 + \left(\frac{9}{2}-\frac{18}{5}-6+\frac{36}{7}+\frac{9}{4}-2\right)\lambda^2$$

$$+\left(\frac{9}{2}-\frac{36}{5}+\frac{36}{7}-\frac{9}{4}\right)\lambda + \left(\frac{3}{2}-\frac{18}{5}+3-\frac{6}{7}\right)$$

$$A_2 = 0.15000\lambda^3 + 0.292857\lambda^2 + 0.192857\lambda + 0.042857$$

$$A_1 = -\frac{2}{3}(\lambda^2+2\lambda+1) + (1+\lambda) + \frac{2}{5}[4\lambda^2+7\lambda+2] - \frac{2}{3}(3+4\lambda) - \frac{2}{7}(7\lambda^2+6\lambda-3)$$

$$+\frac{3}{2}\lambda + \frac{2}{3}\lambda^2$$

$$= \left(-\frac{2}{3}+\frac{8}{5}-2+\frac{2}{3}\right)\lambda^2 + \left(-\frac{4}{3}+1+\frac{14}{5}-\frac{8}{3}-\frac{12}{7}+\frac{3}{2}\right)\lambda + \left(-\frac{2}{3}+1+\frac{4}{5}-2+\frac{6}{7}\right)$$

$$A_1 = -0.4000\lambda^2 - 0.41429\lambda - 0.009524$$

①	②	③	④	⑤	⑥	⑦	⑧
λ	λ^2	λ^3	λ^4	$0.07389\lambda^4$	$0.20635\lambda^3$	$0.21904\lambda^2$	0.10476λ
−0.5	0.25	−0.125	+0.0625	+0.004618	−0.025794	0.054760	−0.052380
−0.4	0.16	−0.064	+0.0256	+0.001892	−0.013206	0.035046	−0.041904
−0.3	0.09	−0.027	+0.0081	+0.000599	−0.005571	0.019714	−0.031428
−0.2	0.04	−0.008	+0.0016	+0.000118	−0.001651	0.008762	−0.020952
−0.1	0.01	−0.001	+0.0001	+0.000007	−0.000206	0.002190	−0.010476
0	0	0	0	0	0	0	0
+0.1	0.01	+0.001	+0.0001	+0.000007	+0.000206	0.002190	+0.010476
+0.2	0.04	+0.008	+0.0016	+0.000118	+0.001651	0.008762	+0.020952
+0.3	0.09	+0.027	+0.0081	+0.000599	+0.005571	0.019714	+0.031428
+0.4	0.16	+0.064	+0.0256	+0.001892	+0.013206	0.035046	+0.041904
+0.5	0.25	+0.125	+0.0625	+0.004618	+0.025794	0.054760	+0.052380

	⑨	⑩	⑪	⑫	⑬	⑭	⑮
λ	A_3	$0.15000\lambda^3$	0.29[illegible]λ^2	0.1928[illegible]λ	A_2	$-0.40000\lambda^2$	-0.41429λ
−0.5	+0.000254	−0.01875	0.0[illegible]14	−0.[illegible]	+0.017767	−0.100000	+0.207145
−0.4	+0.000878	−0.00960	0.046857	−0.[illegible]43	+0.011611	−0.064000	+0.165716
−0.3	+0.002364	−0.00405	0.026357	−0.057857	+0.010952	−0.036000	+0.124287
−0.2	+0.005327	−0.00120	0.011714	−0.038571	+0.015880	−0.016000	+0.082858
−0.1	+0.010565	−0.00015	0.002929	−0.019286	+0.026485	−0.004000	+0.041429
0	+0.019050	0	0	0	+0.042857	0	0
+0.1	+0.031929	+0.00015	0.002929	+0.019286	+0.065087	−0.004000	−0.041429
+0.2	+0.050533	+0.00120	0.011714	+0.038571	+0.093262	−0.016000	−0.082858
+0.3	+0.076362	+0.00405	0.026357	+0.057857	+0.127476	−0.036000	−0.124287
+0.4	+0.111098	+0.00960	0.046857	+0.[illegible]7143	+0.167817	−0.064000	−0.165716
+0.5	+0.156602	+0.01875	0.073214	+0.096429	+0.214375	−0.100000	−0.207145

	⑯	⑰	⑱	⑲	⑳	㉑	㉒
λ	A_1	A_0	B_1	-2λ	B_0	$k_1=4A_3B_1$	B_1A_2
-0.5	+0.097621	-0.010714	0.133333	+1.00	-0.33333	+0.0001355	+0.0023689
-0.4	+0.092192	-0.019048	0.245333	+0.8	-0.53333	+0.0008616	+0.0028486
-0.3	+0.078763	-0.027381	0.421333	+0.6	-0.73333	+0.003984	+0.0046144
-0.2	+0.057334	-0.035714	0.661333	+0.4	-0.93333	+0.014092	+0.0105020
-0.1	+0.027905	-0.044048	0.965333	+0.2	-1.13333	+0.040795	+0.0255668
0	-0.009524	-0.052381	1.33333	0	-1.33333	+0.101600	+0.057143
+0.1	-0.054953	-0.060714	1.765333	-0.2	-1.53333	+0.225461	+0.114900
+0.2	-0.108382	-0.069048	2.261333	-0.4	-1.73333	+0.457088	+0.210896
+0.3	-0.169811	-0.077381	2.821333	-0.6	-1.93333	+0.861772	+0.359652
+0.4	-0.239240	-0.085714	3.445333	-0.8	-2.13333	+1.53108	+0.578185
+0.5	-0.316669	-0.094048	4.133333	-1.0	-2.33333	+2.58915	+0.886083

	㉓	㉔	㉕	㉖	㉗	㉘	㉙
λ	B_0A_3	K_2	A_2B_0	B_1A_1	K_3	A_1B_0	B_1A_0
-0.5	-0.0000847	+0.0068526	-0.005922	+0.013016	+0.014188	-0.032540	-0.001429
-0.4	-0.0004683	+0.0071409	-0.006192	+0.022618	+0.039852	-0.049169	-0.004673
-0.3	-0.0017336	+0.0086424	-0.008031	+0.033185	+0.050308	-0.057759	-0.011537
-0.2	-0.0049718	+0.0165906	-0.014821	+0.037917	+0.046192	-0.053512	-0.023619
-0.1	-0.0119736	+0.0407796	-0.030016	+0.026938	-0.006156	-0.031626	-0.042521
0	-0.0253999	+0.095229	-0.0571425	-0.012699	-0.139684	+0.012699	-0.069841
+0.1	-0.0489577	+0.197826	-0.0997998	-0.097010	-0.393820	+0.0842611	-0.107180
+0.2	-0.0875904	+0.369918	-0.161654	-0.245088	-0.813484	+0.187862	-0.156141
+0.3	-0.147633	+0.636057	-0.246453	-0.479093	-1.451092	+0.328301	-0.218318
+0.4	-0.237009	+1.023528	-0.358009	-0.824261	-2.364510	+0.510378	-0.295313
+0.5	-0.365404	+1.562037	-0.500208	-1.30890	-3.618216	+0.738893	-0.388732

$$C_1^3 + \alpha_1 C_1^2 + \alpha_2 C_1 + \alpha_3 = 0$$

	(30)	(31)	(32)	(33)	(34)	(35)	(36)
λ	K_4	α_1	α_2	α_3	$\xi = -\frac{\alpha_1}{3}$	$-\alpha_1^2/3$	p
−0.5	−0.033969	+50.573	+104.71	−250.69	−16.858	−852.56	−747.85
−0.4	−0.053842	+8.2879	+38.129	−62.491	−2.7626	−22.896	+15.233
−0.3	−0.069296	+2.1692	+12.627	−17.393	−0.72307	−1.5685	+11.058
−0.2	−0.077131	+1.1773	+3.2779	−5.4734	−0.39243	−0.46201	+2.8159
−0.1	−0.074147	+0.99962	−0.1509	−1.8176	−0.33321	−0.33308	−0.48398
0	−0.057142	+0.93729	−1.3748	−0.56242	—	—	—
+0.1	−0.022419	+0.87743	−1.7467	−0.10165	−0.29248	−0.25663	−2.0033
+0.2	+0.031721	+0.80929	−1.7797	+0.069398	−0.26976	−0.21831	−1.9980
+0.3	+0.109983	+0.73808	−1.6838	+0.12762	−0.24603	−0.18159	−1.8654
+0.4	+0.215065	+0.66850	−1.5444	+0.14047	−0.22283	−0.14896	−1.6934
+0.5	+0.350161	+0.60330	−1.3975	+0.13524	−0.20110	−0.12132	−1.5188

	(37)	(38)	(39)	(40)	(41)	(42)	(43)
λ	$\frac{2}{9}\alpha_1^2$	(37)−(32)	−ξ (38)	q	$-\frac{p}{3}$	r	r^3
−0.5	568.37	+463.66	7816.4	7565.7			
−0.4	15.264	−22.865	−63.167	−125.66			
−0.3	1.0457	−11.581	−8.3739	−25.767			
−0.2	0.30801	−2.9699	−1.1655	−6.6389			
−0.1	0.22205	+0.37295	+0.12427	−1.6933	+0.16133		
0	—	—	—	—	—		
+0.1	0.17109	+1.9178	+0.56092	+0.45927	+0.66777	+0.81717	+0.54568
+0.2	0.14554	+1.9252	+0.51934	+0.58874	+0.66600	+0.81609	+0.54352
+0.3	0.12106	+1.8049	+0.44406	+0.57168	+0.62180	+0.78854	+0.49031
+0.4	0.09931	+1.6437	+0.36627	+0.50674	+0.56447	+0.75132	+0.42410
+0.5	0.08088	+1.4784	+0.29731	+0.43255	+0.50627	+0.71153	+0.36023

	(44)	(45)	(46)	(47)	(48)	(49)
λ	$q/2$	$\cos\theta$	θ°	β_1	β_2	β_3
+0.1	+0.22964	−0.42083	114° 53.2′	38° 17.7′	158° 17.7′	−81° 42.3′
+0.2	+0.29437	−0.54160	122° 47.6′	40° 55.9′	160° 55.9′	−79° 41′
+0.3	+0.28584	−0.58298	125° 39.6′	41° 53.2′	161° 53.2′	−78° 6.8′
+0.4	+0.25337	−0.59743	126° 41.2′	42° 13.7′	162° 13.7′	−77° 46.3′
+0.5	+0.21628	−0.60039	126° 53.9′	42° 18′	162° 18′	−77° 42′

	(50)	(51)	(52)	(53)	(54)	(55)
λ	$\cos\beta_1$	$\cos\beta_2$	$\cos\beta_3$	χ_1	χ_2	χ_3
+0.1	0.78483	−0.92910	0.14945	1.28268	−1.51847	0.24425
+0.2	0.75549	−0.94513	0.18964	1.23310	−1.54262	0.30953
+0.3	0.74447	−0.95045	0.20598	1.17409	−1.49894	0.32485
+0.4	0.74047	−0.95228	0.21181	1.11266	−1.43093	0.31827
+0.5	0.73963	−0.95266	0.21303	1.05254	−1.35519	0.30315

	(56)	(57)	(58)	(59)	(60)	(61)
λ	$C_1^{(1)}$	$C_1^{(2)}$	$C_1^{(3)}$	$B_1C_1^{(1)}$	$B_1C_1^{(2)}$	$B_1C_1^{(3)}$
+0.1	0.99020	−1.81095	−0.04823	1.74803	−3.19692	−0.08514
+0.2	0.96334	−1.81238	+0.03977	2.17843	−4.09839	+0.08993
+0.3	0.92806	−1.74500	+0.07882	2.61836	−4.92322	+0.22238
+0.4	0.88983	−1.65376	+0.09544	3.06576	−5.69775	+0.32882
+0.5	0.85144	−1.55679	+0.10205	3.51928	−6.43473	+0.42181

λ	(59) + B_0	(60) + B_0	(61) + B_0	$2+3\lambda$	$6/(2+3\lambda)$
+0.1	+0.21470	−4.73025	−1.61847	2.3	2.6087
+0.2	+0.44510	−5.83172	−1.64340	2.6	2.3077
+0.3	+0.68503	−6.85655	−1.71095	2.9	2.0690
+0.4	+0.93243	−7.83108	−1.80451	3.2	1.8750
+0.5	+1.18595	−8.76806	−1.91152	3.5	1.7143

X = No Use

	(62)	(63)	(64)	(65)	(66)	(67)
λ	$C_1^{(1)2}$	$C_1^{(2)2}$	$C_1^{(3)2}$	$A_3 C_1^{(1)2}$	$\frac{2}{3} A_2 C_1^{(1)}$	(65) + (66) + $\frac{A_1}{3}$
+0.1	0.98050	3.2795	0.002326	0.031306	0.042966	0.055954
0.2	0.92802	3.2847	0.001589	0.046896	0.059896	0.070665
0.3	0.86130	3.0450	0.006213	0.065771	0.078870	0.088037
0.4	0.79180	2.7349	0.009109	0.087967	0.099552	0.107772
0.5	0.72495	2.4236	0.010414	0.113529	0.121685	0.129658

	(68)	(69)	(70)	(71)	(72)	(73)
λ	[ ]	$A_3 C_1^{(2)2}$	$\frac{2}{3} A_2 C_1^{(2)}$	(69) + (70) + $\frac{A_1}{3}$	[ ]	$A_3 C_1^{(3)2}$
+0.1	0.037292	0.10471	-0.07858	0.007812	0.005206	0.00007427
0.2		0.16599	-0.11268	0.017183		0.00007994
0.3		0.23252	-0.14830	0.027580		0.00047444
0.4		0.30384	-0.18502	0.039073		0.00101199
0.5		0.37954	-0.22249	0.051494		0.0017779

	(74)	(75)	(76)	(77)	(78)	(79)
λ	$\frac{2}{3} A_2 C_1^{(3)}$	- (73) + (74) + $\frac{A_1}{3}$	[ ]	$(76)^{\frac{1}{2}}$	ϕ	
0.1	-0.002093	0.020337	0.011520	0.10733	0.4532	
0.2	+0.002473	0.033574	0.014847	0.12185	0.4621	
0.3	+0.006698	0.049432	0.017521	0.13237	0.4686	
0.4	+0.010678	0.068057	0.019753	0.14054	0.4755	
0.5	+0.014585	0.089193	0.021579	0.14690	0.4814	

	①	②	③	④	⑤	⑥
λ	$p/3$	$①^3$	$q/2$	$③^2$	④+②	$⑤^{\frac{1}{2}}$
−0.5	−249.28	−15,491,000	3782.9	14,310,000	—	—
−0.4	+5.0777	+130.92	−62.830	+3947.6	4078.5	63.863
−0.3	+3.6860	+50.082	−12.884	+166.00	216.08	14.700
−0.2	+0.93863	+0.82696	−3.3195	+11.019	11.846	3.4418
−0.1	−0.16133	−0.004199	−0.84665	+0.71682	0.71262	0.84417
	⑦	⑧	⑨	⑩	⑪	⑫
λ	A	B	$(A)^{\frac{1}{3}}$	$B^{\frac{1}{3}}$	x_1	C_1
−0.5	—	—	—	—		
−0.4	126.693	−1.033	5.0225	−1.011	4.012	1.249
−0.3	27.584	−1.816	3.0215	−1.222	1.800	1.077
−0.2	6.7613	−0.1223	1.8909	−0.496	1.395	1.003
−0.1	1.69082	+0.00248	1.1913	+0.1354	1.3267	1.000

If $\Theta = C\left[\alpha + \left(\frac{1}{C}-1\right)\frac{\alpha^5}{\beta^4}\right]$

$$\Theta^2 - \alpha^2 = \alpha^2\left[(C^2-1) + 2C^2\left(\frac{1}{C}-1\right)\left(\frac{\alpha}{\beta}\right)^4 + C^2\left(\frac{1}{C}-1\right)^2\left(\frac{\alpha}{\beta}\right)^8\right]$$

$$\frac{d\Theta}{d\alpha} - 1 = (C-1)\left\{1 - 5\left(\frac{\alpha}{\beta}\right)^4\right\}$$

$$\frac{\Theta}{\alpha} - 1 = (C-1)\left\{1 - \left(\frac{\alpha}{\beta}\right)^4\right\}$$

The total potential energy

$$\frac{E\left(\frac{t}{R}\right)}{4}\int_0^\beta \Big\{(C^2-1)^2 + 4C^2\left(\frac{1}{C}-1\right)(C^2-1)\left(\frac{\alpha}{\beta}\right)^4 + 2C^2(3C^2-1)\left(\frac{1}{C}-1\right)^2\left(\frac{\alpha}{\beta}\right)^8 + 4C^4\left(\frac{1}{C}-1\right)^3\left(\frac{\alpha}{\beta}\right)^{12}$$
$$+ C^4\left(\frac{1}{C}-1\right)^4\left(\frac{\alpha}{\beta}\right)^{16}\Big\}\alpha^5\,d\alpha$$
$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)^2\int_0^\beta\left\{2 - 12\left(\frac{\alpha}{\beta}\right)^4 + 26\left(\frac{\alpha}{\beta}\right)^8\right\}\alpha\,d\alpha + pC\int_0^\beta\left\{\alpha^3 + \left(\frac{1}{C}-1\right)\frac{\alpha^7}{\beta^4}\right\}d\alpha$$

$$= \frac{E\frac{t}{R}}{4}\Big\{\frac{(C^2-1)^2}{6} + \frac{2}{5}C^2\left(\frac{1}{C}-1\right)(C^2-1) + \frac{1}{7}C^2(3C^2-1)\left(\frac{1}{C}-1\right)^2 + \frac{2}{9}C^4\left(\frac{1}{C}-1\right)^3$$
$$+ \frac{1}{22}C^4\left(\frac{1}{C}-1\right)^4\Big\}\beta^6$$
$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)^2\frac{16}{10}\beta^2 + pC\left\{\frac{1}{4} + \frac{1}{8}\left(\frac{1}{C}-1\right)\right\}\beta^4$$

Minimizing with respect to C

$$\frac{E\left(\frac{t}{R}\right)}{4}\Big\{\frac{2}{3}C(C^2-1) - \frac{2}{5}(C-1)(4C^2+C-1) + \frac{2}{7}(C-1)(6C^2-3C-1) - \frac{2}{9}(C-1)(4C^2-5C+1)$$
$$+ \frac{2}{11}(C-1)^3\Big\}\beta^6$$
$$+ \frac{E\left(\frac{t}{R}\right)^3}{12}(C-1)\frac{32}{10}\beta^2 + p\frac{1}{8}\beta^4 = 0.$$

$$p = 2E\left(\frac{t}{R}\right)(1-c)\left\{\beta^2\left[\frac{2}{3}c(c+1) - \frac{2}{5}(4c^2+c-1) + \frac{2}{7}(6c^2-3c-1) - \frac{2}{9}(4c^2-5c+1) + \frac{2}{11}(c-1)^2\right\} + \frac{32}{30}\left(\frac{t}{R}\right)^2\beta^{-2}\right\}$$

$$\beta^2 = \frac{4}{\sqrt{15}}\left(\frac{t}{R}\right)\left[\quad\right]^{-\frac{1}{2}}$$

$$\frac{p}{2E\left(\frac{t}{R}\right)^2} = \phi = \frac{8}{\sqrt{15}}(1-c)\left[\frac{2}{3}(c^2+c) - \frac{2}{5}(4c^2+c-1) + \frac{2}{7}(6c^2-3c-1) - \frac{2}{9}(4c^2-5c+1) + \frac{2}{11}(c-1)^2\right]^{\frac{1}{2}}$$

$$= \frac{8\sqrt{2}}{\sqrt{15}}(1-c)\frac{1}{3\sqrt{385}}\left[1155(c^2+c) - 693(4c^2+c-1) + 495(6c^2-3c-1) - 385(4c^2-5c+1) + 315(c^2-2c+1)\right]^{\frac{1}{2}}$$

$$= \frac{8}{15}\sqrt{2}\frac{1}{\sqrt{231}}(1-c)\left[128c^2 + 272c + 128\right]^{\frac{1}{2}}$$

$$= \frac{16\sqrt{2}}{15\sqrt{231}}(1-c)\left[32c^2 + 68c + 32\right]^{\frac{1}{2}}$$

$$= \frac{32\sqrt{2}}{15\sqrt{231}}(1-c)\left[8c^2 + 17c + 8\right]^{\frac{1}{2}}$$

$$(1-c)^2\left[8c^2+17c+8\right] = (c-1)^2\left[8c^2+17c+8\right]$$

$$+2(8c^2+17c+8) + (c-1)(16c+17)$$

$$16c^2+34c+16$$
$$16c^2+17c$$
$$-16c-17$$

$$32c^2+35c-1$$

$$c = \frac{1}{64}\left[-35 \pm \sqrt{1225+128}\right] \qquad \begin{array}{r}128\\ \hline 1353\end{array}$$

$$= \frac{1}{64}\left[-35 \pm 36.783\right] = \begin{array}{l} + \frac{1.783}{64} = 0.025643 \\ - \frac{71.783}{64} \quad -1.1216 \end{array}$$

$$8c^2+17c+8 = (1-c)\,\frac{(16c+17)}{2} = (1-c)\ \begin{array}{l}8.7229\\ -0.4729\end{array} \text{ Impo.}$$

$$= 0.974357 \times 8.7229$$

$$\phi = 0.974357^{3/2} \times \left(\frac{2248 \times 8.7229}{51975}\right)^{\frac{1}{2}} = 0.59075$$

Let $\theta = c_1\left[\alpha + \left(\frac{1}{c_1}-1\right)\frac{\alpha^3}{\beta^2}\right] + c_2\alpha\left[1-\left(\frac{\alpha}{\beta}\right)^4\right]$ (1)

$$= \alpha\left\{(c_1+c_2) + (1-c_1)\left(\frac{\alpha}{\beta}\right)^2 - c_2\left(\frac{\alpha}{\beta}\right)^4\right\}$$

$$\theta^2 = \alpha^2\left\{(c_1+c_2)^2 + 2(c_1+c_2)(1-c_1)\left(\frac{\alpha}{\beta}\right)^2 + \left[(1-c_1)^2 - 2c_2(c_1+c_2)\right]\left(\frac{\alpha}{\beta}\right)^4 - 2c_2(1-c_1)\left(\frac{\alpha}{\beta}\right)^6 + c_2^2\left(\frac{\alpha}{\beta}\right)^8\right\}$$

$$\theta^2-\alpha^2 = \alpha^2\left\{\left[(c_1+c_2)^2-1\right] + 2(c_1+c_2)(1-c_1)\left(\frac{\alpha}{\beta}\right)^2 + \left[(1-c_1)^2 - 2c_2(c_1+c_2)\right]\left(\frac{\alpha}{\beta}\right)^4 - 2c_2(1-c_1)\left(\frac{\alpha}{\beta}\right)^6 + c_2^2\left(\frac{\alpha}{\beta}\right)^8\right\}$$

$$(\theta^2-\alpha^2)^2 = \alpha^4\left\{\left[(c_1+c_2)^2-1\right]^2 + 4(c_1+c_2)(1-c_1)\left[(c_1+c_2)^2-1\right]\left(\frac{\alpha}{\beta}\right)^2\right.$$

$$+\left[4(c_1+c_2)^2(1-c_1)^2 + 2\{(c_1+c_2)^2-1\}\{(1-c_1)^2 - 2c_2(c_1+c_2)\}\right]\left(\frac{\alpha}{\beta}\right)^4$$

$$+\left[4(c_1+c_2)(1-c_1)\{(1-c_1)^2 - 2c_2(c_1+c_2)\} - 4c_2(1-c_1)\{(c_1+c_2)^2-1\}\right]\left(\frac{\alpha}{\beta}\right)^6$$

$$+\left[\{(1-c_1)^2 - 2c_2(c_1+c_2)\}^2 - 8c_2(1-c_1)^2(c_1+c_2) + 2c_2^2\{(c_1+c_2)^2-1\}\right]\left(\frac{\alpha}{\beta}\right)^8$$

$$+\left[4c_2^2(c_1+c_2)(1-c_1) - 4c_2(1-c_1)\{(1-c_1)^2 - 2c_2(c_1+c_2)\}\right]\left(\frac{\alpha}{\beta}\right)^{10}$$

$$+\left[4c_2^2(1-c_1)^2 + 2c_2^2\{(1-c_1)^2 - 2c_2(c_1+c_2)\}\right]\left(\frac{\alpha}{\beta}\right)^{12}$$

(2)

$$+\left[-4C_2^3(1-C_1)\right]\left(\frac{\alpha}{\beta}\right)^{14}+C_2^4\left(\frac{\alpha}{\beta}\right)^{16}$$

$$\int_0^\beta(\theta^2-\alpha^2)^2\alpha\,d\alpha=\beta^6\Big\{\frac{1}{6}\left[(C_1+C_2)^2-1\right]^2+\frac{1}{2}(C_1+C_2)(1-C_1)\left[(C_1+C_2)^2-1\right]$$

$$+\frac{2}{5}(C_1+C_2)^2(1-C_1)^2+\frac{1}{5}\left[(C_1+C_2)^2-1\right]\left[(1-C_1)^2-2C_2(C_1+C_2)\right]$$

$$+\frac{1}{3}(C_1+C_2)(1-C_1)\left[(1-C_1)^2-2C_2(C_1+C_2)\right]-\frac{1}{3}C_2(1-C_1)\left[(C_1+C_2)^2-1\right]$$

$$+\frac{1}{14}\left[(1-C_1)^2-2C_2(C_1+C_2)\right]^2-\frac{4}{7}C_2(1-C_1)^2(C_1+C_2)+\frac{1}{7}C_2^2\left[(C_1+C_2)^2-1\right]$$

$$+\frac{1}{4}C_2^2(C_1+C_2)(1-C_1)-\frac{1}{4}C_2(1-C_1)\left[(1-C_1)^2-2C_2(C_1+C_2)\right]$$

$$+\frac{2}{9}C_2^2(1-C_1)^2+\frac{1}{9}C_2^2\left[(1-C_1)^2-2C_2(C_1+C_2)\right]-\frac{1}{5}C_2^3(1-C_1)$$

$$+\frac{1}{22}C_2^4\Big\}$$

put $(C_1+C_2)=a$, $(1-C_1)=b$, $C_2=c$

$$=\frac{1}{6}(a^2-1)^2+\frac{1}{2}ab(a^2-1)+\frac{2}{5}a^2b^2+\frac{1}{5}(a^2-1)(b^2-2ac)$$

$$+\frac{1}{3}ab(b^2-2ac)-\frac{1}{3}cb(a^2-1)+\frac{1}{14}(b^2-2ac)^2-\frac{4}{7}cb^2a$$

$$+\frac{1}{7}c^2(a^2-1)+\frac{1}{4}c^2ab-\frac{1}{4}cb(b^2-2ac)+\frac{2}{9}c^2b^2+\frac{1}{9}c^2(b^2-2ac)$$

$$-\frac{1}{5}c^3b+\frac{1}{22}c^4$$

$$= \frac{1}{6}(a^4 - 2a^2 + 1) + \frac{1}{2}(a^3b - ab) + \frac{2}{5}a^2b^2 + \frac{1}{5}(a^2b^2 - 2a^3c - b^2 + 2ac) \quad (3)$$

$$+ \frac{1}{3}(ab^3 - 2a^2bc) - \frac{1}{3}(a^2bc - bc) + \frac{1}{14}(b^4 - 4ab^2c + 4a^2c^2)$$

$$- \frac{4}{7}ab^2c - \frac{1}{7}(a^2c^2 - c^2) + \frac{1}{4}abc^2 - \frac{1}{4}(b^3c - 2abc^2) + \frac{2}{9}b^2c^2$$

$$+ \frac{1}{9}(b^2c^2 - 2ac^3) - \frac{1}{5}bc^3 + \frac{1}{22}c^4$$

$$= a^4\left(\frac{1}{6}\right) + a^3b\left(\frac{1}{2}\right) + a^2b^2\left(\frac{2}{5} + \frac{1}{5}\right) + ab^3\left(\frac{1}{3}\right) + b^4\left(\frac{1}{14}\right)$$

$$+ a^3c\left(-\frac{2}{5}\right) + a^2c^2\left(\frac{2}{7} - \frac{1}{7}\right) + ac^3\left(-\frac{2}{9}\right) + c^4\left(\frac{1}{22}\right)$$

$$+ b^3c\left(-\frac{1}{4}\right) + b^2c^2\left(\frac{2}{9} + \frac{1}{9}\right) + bc^3\left(-\frac{1}{5}\right) + a^2bc\left(-\frac{2}{3} - \frac{1}{3}\right)$$

$$+ ab^2c\left(-\frac{2}{7} - \frac{4}{7}\right) + abc^2\left(\frac{1}{4} + \frac{1}{2}\right) - \frac{1}{3}a^2 + \frac{1}{6} - \frac{1}{2}ab$$

$$- \frac{1}{5}b^2 + \frac{2}{5}ac + \frac{1}{3}bc + \frac{1}{7}c^2$$

$$= \frac{1}{6}a^4 + \frac{1}{2}a^3b + \frac{3}{5}a^2b^2 + \frac{1}{3}ab^3 + \frac{1}{14}b^4 - \frac{2}{5}a^3c + \frac{1}{7}a^2c^2$$

$$- \frac{2}{9}ac^3 + \frac{1}{22}c^4 - \frac{1}{4}b^3c + \frac{1}{3}b^2c^2 - \frac{1}{5}bc^3 - a^2bc - \frac{6}{7}ab^2c$$

$$+ \frac{3}{4}abc^2 - \frac{1}{3}a^2 - \frac{1}{2}ab - \frac{1}{5}b^2 + \frac{2}{5}ac + \frac{1}{3}bc + \frac{1}{7}c^2 + \frac{1}{6}$$

(4)

$$= \frac{1}{6}\left(C_1^4 + 4C_1^3C_2 + 6C_1^2C_2^2 + 4C_1C_2^3 + C_2^4\right)$$

$$+\frac{1}{2}\left(C_1^3 + 3C_1^2C_2 + 3C_1C_2^2 + C_2^3 - C_1^4 - 3C_1^3C_2 - 3C_1^2C_2^2 - C_1C_2^3\right)$$

$$+\frac{3}{5}\left(C_1^4 + 2C_1^3C_2 + C_1^2C_2^2 - 2C_1^3 - 4C_1^2C_2 - 2C_1C_2^2 + C_1^2 + 2C_1C_2 + C_2^2\right)$$

$$+\frac{1}{3}\left(C_1 - 3C_1^2 + 3C_1^3 - C_1^4 + C_2 - 3C_1C_2 + 3C_1^2C_2 - C_1^3C_2\right)$$

$$+\frac{1}{14}\left(1 - 4C_1 + 6C_1^2 - 4C_1^3 + C_1^4\right)$$

$$-\frac{2}{5}\left(C_1^3C_2 + 3C_1^2C_2^2 + 3C_1C_2^3 + C_2^4\right)$$

$$+\frac{1}{7}\left(C_1^2C_2^2 + 2C_1C_2^3 + C_2^4\right)$$

$$-\frac{2}{9}\left(C_1C_2^3 + C_2^4\right) + \frac{1}{22}C_2^4$$

$$-\frac{1}{4}\left(C_2 - 3C_1C_2 + 3C_1^2C_2 - C_1^3C_2\right)$$

$$+\frac{1}{3}\left(C_2^2 - 2C_1C_2^2 + C_1^2C_2^2\right) - \frac{1}{5}\left(C_2^3 - C_1C_2^3\right)$$

$$-\left(C_1^2C_2 + 2C_1C_2^2 + C_2^3 - C_1^3C_2 - 2C_1^2C_2^2 - C_1C_2^3\right)$$

$$-\frac{6}{7}\left(C_1C_2 - 2C_1^2C_2 + C_1^3C_2 + C_2^2 - 2C_1C_2^2 + C_1^2C_2^2\right)$$

$$+\frac{3}{4}\left(-C_1^2C_2^2 - C_1C_2^3 + C_1C_2^2 + C_2^3\right) - \frac{1}{3}\left(C_1^2 + 2C_1C_2 + C_2^2\right)$$

$$-\frac{1}{2}\left(-C_1^2 - C_1C_2 + C_1 + C_2\right) - \frac{1}{5}\left(1 - 2C_1 + C_1^2\right)$$

$$+\frac{9}{5}\left(C_1C_2+C_2^2\right)+\frac{1}{3}\left(C_2-C_1C_2\right)+\frac{1}{7}C_2^2+\frac{1}{6} \qquad (5)$$

$$= C_1^4\left(\frac{1}{6}-\frac{1}{2}+\frac{3}{5}-\frac{1}{3}+\frac{1}{14}\right)+C_1^3C_2\left(\frac{2}{3}-\frac{3}{2}+\frac{6}{5}-\frac{1}{3}-\frac{2}{5}+\frac{1}{4}+1-\frac{1}{7}\right)$$

$$+C_1^2C_2^2\left(1-\frac{3}{2}+\frac{3}{5}-\frac{6}{5}+\frac{1}{7}+\frac{1}{3}+2-\frac{6}{7}-\frac{3}{4}\right)$$

$$+C_1C_2^3\left(\frac{2}{3}-\frac{1}{2}-\frac{6}{5}+\frac{9}{7}-\frac{2}{9}+\frac{1}{5}+1-\frac{3}{4}\right)$$

$$+C_2^4\left(\frac{1}{6}-\frac{2}{5}+\frac{1}{7}+\frac{1}{22}-\frac{2}{9}\right)$$

$$+C_1^3\left(\frac{1}{2}-\frac{6}{5}+1-\frac{2}{7}\right)$$

85 terms

$$+C_1^2C_2\left(\frac{3}{2}-\frac{12}{5}+1-\frac{3}{4}-1+\frac{12}{7}\right)$$

$$+C_1C_2^2\left(\frac{3}{2}-\frac{6}{5}-\frac{2}{3}-2+\frac{10}{7}+\frac{3}{4}\right)$$

$$+C_2^3\left(\frac{1}{2}-\frac{1}{5}-1+\frac{9}{4}\right)$$

$$+C_1^2\left(\frac{3}{5}-1+\frac{3}{7}-\frac{1}{3}+\frac{1}{2}-\frac{1}{5}\right)$$

$$+C_1C_2\left(\frac{6}{5}-1+\frac{3}{4}-\frac{6}{7}-\frac{2}{3}+\frac{1}{2}+\frac{2}{5}-\frac{1}{3}\right)$$

$$+C_2^2\left(\frac{3}{5}+\frac{1}{3}-\frac{6}{7}-\frac{1}{3}+\frac{2}{5}+\frac{1}{7}\right)$$

$$+C_1\left(\frac{1}{3}-\frac{2}{7}-\frac{1}{2}+\frac{2}{5}\right)+C_2\left(-\frac{1}{4}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}\right)$$

$$+\left(\frac{1}{14}-\frac{1}{5}+\frac{1}{6}\right)$$

$$\int_0^\beta (\theta^2-\alpha^2)^2 \alpha\, d\alpha = \beta^6 \Big[\frac{1}{210} C_1^4 + \frac{11}{420} C_1^3 C_2 - \frac{97}{420} C_1^2 C_2^2 \qquad (6)$$

$$- \frac{131}{252} C_1 C_2^3 - \frac{926}{3465} C_2^4 + \frac{1}{70} C_1^3 + \frac{9}{140} C_1^2 C_2 + \frac{41}{420} C_1 C_2^2$$

$$+ \frac{1}{20} C_2^3 - \frac{1}{210} C_1^2 - \frac{1}{140} C_1 C_2 + \frac{2}{7} C_2^2 + \frac{11}{210} C_1 - \frac{1}{12} C_2 + \frac{1}{210} \Big]$$

$$\frac{d\theta}{d\alpha} = (C_1+C_2) + 3(1-C_1)\left(\frac{\alpha}{\beta}\right)^2 - 5C_2\left(\frac{\alpha}{\beta}\right)^4$$

$$\left(\frac{d\theta}{d\alpha}-1\right)^2 = (C_1+C_2-1)^2 + 6(1-C_1)(C_1+C_2-1)\left(\frac{\alpha}{\beta}\right)^2$$

$$+ \left[9(1-C_1)^2 - 10C_2(C_1+C_2-1)\right]\left(\frac{\alpha}{\beta}\right)^4 - 30C_2(1-C_1)\left(\frac{\alpha}{\beta}\right)^6 + 25C_2^2\left(\frac{\alpha}{\beta}\right)^8$$

$$\left(\frac{\theta}{\alpha}-1\right)^2 = (C_1+C_2-1)^2 + 2(1-C_1)(C_1+C_2-1)\left(\frac{\alpha}{\beta}\right)^2$$

$$+ \left[(1-C_1)^2 - C_2(C_1+C_2-1)\right]\left(\frac{\alpha}{\beta}\right)^4 - 2C_2(1-C_1)\left(\frac{\alpha}{\beta}\right)^6 + C_2^2\left(\frac{\alpha}{\beta}\right)^8$$

$$\int_0^\beta \left\{ \left(\frac{d\theta}{d\alpha}-1\right)^2 + \left(\frac{\theta}{\alpha}-1\right)^2 \right\} \alpha\, d\alpha$$

$$= \beta^2 \Big[(C_1+C_2-1)^2 + 2(1-C_1)(C_1+C_2-1) + \frac{5}{3}(1-C_1)^2 - 2C_2(C_1+C_2-1)$$

$$- 4C_2(1-C_1) + 2.6\, C_2^2 \Big]$$

$$\int_0^\beta \Theta\alpha^2 d\alpha = \beta^4\left[\frac{1}{4}(C_1+C_2)+\frac{1}{6}(1-C_1)-\frac{1}{8}C_2\right]$$

$$-\beta^4\left[\frac{1}{6}+\frac{1}{12}C_1+\frac{1}{8}C_2\right]$$

$$\int_0^\beta \left\{\left(\frac{d\Theta}{d\alpha}-1\right)^2+\left(\frac{\Theta}{\alpha}-1\right)^2\right\}\alpha\, d\alpha$$

$$=\beta^2\left[\frac{2}{3}C_1^2+2C_1C_2+1.6\,C_2^2-\frac{4}{3}C_1-2C_2+\frac{2}{3}\right]$$

$$z_0 = R\int_0^\beta \Theta\, d\alpha = R\beta^2\left[\frac{1}{2}(C_1+C_2)+\frac{1}{4}(1-C_1)-\frac{1}{6}C_2\right]$$

$$= R\beta^2\left[\frac{1+C_1}{4}+\frac{1}{3}C_2\right]$$

$$\delta = R\beta^2\left[\frac{1-C_1}{4}-\frac{1}{3}C_2\right]$$

$$\frac{\delta}{t} = w = \left(\frac{R}{t}\right)\beta^2\left[\frac{1}{4}(1-C_1)-\frac{1}{3}C_2\right]$$

the total energy

$$E\left(\tfrac{t}{R}\right)\beta^6\Big\{\tfrac{1}{210}C_1^4+\tfrac{11}{420}C_1^3C_2-\tfrac{97}{420}C_1^2C_2^2-\tfrac{131}{252}C_1C_2^3-\tfrac{926}{3465}C_2^4+\tfrac{1}{70}C_1^3+\tfrac{9}{140}C_1^2C_2$$

$$+\tfrac{41}{420}C_1C_2^2+\tfrac{1}{20}C_2^3-\tfrac{1}{210}C_1^2-\tfrac{1}{140}C_1C_2+\tfrac{2}{7}C_2^2-\tfrac{11}{210}C_1-\tfrac{1}{12}C_2+\tfrac{8}{210}\Big\}$$

$$+E\left(\tfrac{t}{R}\right)^3\beta^2\Big\{\tfrac{2}{9}C_1^2+\tfrac{2}{3}C_1C_2+\tfrac{8}{15}C_2^2-\tfrac{4}{9}C_1-\tfrac{2}{3}C_2+\tfrac{2}{9}\Big\}$$

$$+\,p\beta^4\Big\{\tfrac{2}{3}+\tfrac{1}{3}C_1+\tfrac{1}{2}C_2\Big\}$$

Put $\sigma = \frac{p}{2}\frac{1}{\left(\frac{t}{R}\right)}$

$$K=\frac{\sigma}{E\left(\frac{t}{R}\right)}=\frac{p}{2}\frac{1}{E\left(\frac{t}{R}\right)^2}$$

$$\frac{\beta^2}{\left(\frac{t}{R}\right)}=\gamma^2$$

$$\gamma^6\Big\{\tfrac{1}{210}C_1^4+\tfrac{11}{420}C_1^3C_2-\tfrac{97}{420}C_1^2C_2^2-\tfrac{131}{252}C_1C_2^3-\tfrac{926}{3465}C_2^4+\tfrac{1}{70}C_1^3+\tfrac{9}{140}C_1^2C_2$$

$$+\tfrac{41}{420}C_1C_2^2+\tfrac{1}{20}C_2^3-\tfrac{1}{210}C_1^2-\tfrac{1}{140}C_1C_2+\tfrac{2}{7}C_2^2-\tfrac{11}{210}C_1-\tfrac{1}{12}C_2+\tfrac{8}{210}\Big\}$$

$$+\gamma^2\Big\{\tfrac{2}{9}C_1^2+\tfrac{2}{3}C_1C_2+\tfrac{8}{15}C_2^2-\tfrac{4}{9}C_1-\tfrac{2}{3}C_2+\tfrac{2}{9}\Big\}$$

$$+K\gamma^4\Big\{\tfrac{4}{3}+\tfrac{2}{3}C_1+C_2\Big\}$$

$$w=\gamma^2\Big\{\tfrac{1}{4}(1-C_1)-\tfrac{1}{3}C_2\Big\}$$

$$\gamma^4\left\{\frac{2}{105}C_1^3+\frac{11}{140}C_1^2C_2-\frac{97}{210}C_1C_2^2-\frac{131}{252}C_2^3+\frac{3}{70}C_1^2+\frac{9}{70}C_1C_2+\frac{41}{420}C_2^2-\frac{1}{105}C_1-\frac{1}{140}C_2-\frac{11}{210}\right\}+\left\{\frac{4}{9}C_1+\frac{2}{3}C_2-\frac{4}{9}\right\}+K\gamma^2\left(\frac{2}{3}\right)=0 \quad (9)$$

$$\gamma^4\left\{\frac{11}{420}C_1^3-\frac{97}{210}C_1^2C_2-\frac{131}{84}C_1C_2^2-\frac{3704}{3465}C_2^3+\frac{9}{140}C_1^2+\frac{41}{210}C_1C_2+\frac{3}{20}C_2^2-\frac{1}{140}C_1+\frac{4}{7}C_2-\frac{1}{12}\right\}+\left\{\frac{2}{3}C_1+\frac{16}{15}C_2-\frac{2}{3}\right\}+K\gamma^2=0.$$

$$2\gamma^2\left\{\frac{2}{105}C_1^3+\frac{11}{140}C_1^2C_2-\frac{97}{210}C_1C_2^2-\frac{131}{252}C_2^3+\frac{3}{70}C_1^2+\frac{9}{70}C_1C_2+\frac{41}{420}C_2^2-\frac{1}{105}C_1-\frac{1}{140}C_2-\frac{11}{210}\right\}+\frac{2}{3}K$$

$$+\gamma^4\left\{\frac{2}{35}C_1^2\frac{dC_1}{d\gamma}+\frac{11}{140}\left(2C_1C_2\frac{dC_1}{d\gamma}+C_1^2\frac{dC_2}{d\gamma}\right)-\frac{97}{210}\left(C_2^2\frac{dC_1}{d\gamma}+2C_1C_2\frac{dC_2}{d\gamma}\right)\right.$$

$$-\frac{131}{84}C_2^2\frac{dC_2}{d\gamma}+\frac{3}{35}C_1\frac{dC_1}{d\gamma}+\frac{9}{70}\left(C_2\frac{dC_1}{d\gamma}+C_1\frac{dC_2}{d\gamma}\right)+\frac{41}{210}C_2\frac{dC_2}{d\gamma}$$

$$\left.-\frac{1}{105}\frac{dC_1}{d\gamma}-\frac{1}{140}\frac{dC_2}{d\gamma}\right\}+\left\{\frac{4}{9}\frac{dC_1}{d\gamma}+\frac{2}{3}\frac{dC_2}{d\gamma}\right\}=0$$

$$-\frac{2}{\gamma^2}\left\{\frac{4}{9}C_1+\frac{2}{3}C_2-\frac{4}{9}\right\}-\frac{2}{3}K$$

$$+\gamma^4\left\{\left(\frac{2}{35}C_1^2+\frac{11}{70}C_1C_2-\frac{97}{210}C_2^2+\frac{3}{35}C_1+\frac{9}{70}C_2-\frac{1}{105}\right)\frac{dC_1}{d\gamma}\right.$$

$$\left.+\left(\frac{11}{140}C_1^2-\frac{97}{105}C_1C_2-\frac{131}{84}C_2^2+\frac{9}{70}C_1+\frac{41}{210}C_2-\frac{1}{140}\right)\frac{dC_2}{d\gamma}\right\}$$

$$+\frac{4}{9}\frac{dC_1}{d\gamma}+\frac{2}{3}\frac{dC_2}{d\gamma}=0$$

$$-\frac{2}{\gamma^2}\left\{\frac{2}{3}C_1+\frac{16}{15}C_2-\frac{2}{3}\right\}-K$$

$$+\gamma^4\left\{\left(\frac{11}{140}C_1^2-\frac{97}{105}C_1C_2-\frac{131}{84}C_2^2+\frac{9}{70}C_1+\frac{41}{210}C_2-\frac{1}{140}\right)\frac{dC_1}{d\gamma}\right.$$

$$\left.+\left(-\frac{97}{210}C_1^2-\frac{131}{42}C_1C_2-\frac{3704}{1155}C_2^2+\frac{41}{210}C_1+\frac{3}{10}C_2+\frac{4}{7}\right)\frac{dC_2}{d\gamma}\right\}$$

$$+\left\{\frac{2}{3}\frac{dC_1}{d\gamma}+\frac{16}{15}\frac{dC_2}{d\gamma}\right\}=0$$

$$0=\frac{1}{4}-\frac{1}{4}C_1-\frac{1}{3}C_2-\gamma^2\left(\frac{1}{4}\frac{dC_1}{d\gamma^2}+\frac{1}{3}\frac{dC_2}{d\gamma^2}\right)$$

$$\therefore\quad \frac{1}{3}\gamma^2\frac{dC_2}{d\gamma^2}=\left(\frac{1}{4}-\frac{1}{4}C_1-\frac{1}{3}C_2\right)-\frac{1}{4}\gamma^2\frac{dC_1}{d\gamma^2}$$

$$-\frac{2}{\gamma^2}\left\{\frac{4}{9}C_1+\frac{2}{3}C_2-\frac{4}{9}\right\}-\frac{2}{3}K$$

$$+\gamma^2\left(\frac{1}{4}-\frac{1}{4}C_1-\frac{1}{3}C_2\right)\left(\frac{33}{140}C_1^2-\frac{97}{35}C_1C_2-\frac{131}{28}C_2^2+\frac{27}{70}C_1+\frac{41}{70}C_2-\frac{3}{140}\right)$$

$$+\gamma^4\left(-\frac{1}{560}C_1^2+\frac{119}{140}C_1C_2+\frac{1189}{1680}C_2^2-\frac{3}{280}C_1-\frac{1}{56}C_2-\frac{1}{240}\right)\frac{dC_1}{d\gamma^2}$$

$$+\frac{1}{\gamma^2}\left(\frac{1}{2}-\frac{1}{2}C_1-\frac{2}{3}C_2\right)-\frac{1}{18}\frac{dC_1}{d\gamma^2}=0$$

$$-\frac{2}{\gamma^2}\left\{\frac{2}{3}C_1+\frac{16}{15}C_2-\frac{2}{3}\right\}-K$$

$$+\gamma^2\left(\frac{1}{4}-\frac{1}{4}C_1-\frac{1}{3}C_2\right)\left(-\frac{97}{70}C_1^2-\frac{131}{14}C_1C_2-\frac{3704}{385}C_2^2+\frac{41}{70}C_1+\frac{9}{10}C_2+\frac{12}{7}\right)$$

$$+\gamma^4\left(\frac{119}{280}C_1^2+\frac{1189}{840}C_1C_2+\frac{3907}{4620}C_2^2-\frac{1}{56}C_1-\frac{5}{168}C_2-\frac{61}{140}\right)\frac{dC_1}{d\gamma^2}$$

$$+\frac{1}{\gamma^2}\left(\frac{4}{5}-\frac{4}{5}C_1-\frac{16}{15}C_2\right)-\frac{2}{15}\frac{dC_1}{d\gamma^2}=0$$

$$\left(\frac{1}{4}-\frac{1}{4}C_1-\frac{1}{3}C_2\right)\left(\frac{33}{140}C_1^2-\frac{97}{35}C_1C_2-\frac{131}{28}C_2^2+\frac{27}{70}C_1+\frac{41}{70}C_2-\frac{3}{140}\right)$$

$$=\quad+\frac{33}{560}C_1^2-\frac{97}{140}C_1C_2-\frac{131}{112}C_2^2+\frac{27}{280}C_1+\frac{41}{280}C_2-\frac{3}{560}$$

$$-\frac{33}{560}C_1^3+\frac{97}{140}C_1^2C_2+\frac{131}{112}C_1C_2^2+0-\frac{27}{280}C_1^2-\frac{41}{280}C_1C_2+0+\frac{3}{560}C_1+0+0$$

$$-\frac{11}{140}C_1^2C_2+\frac{97}{105}C_1C_2^2+\frac{131}{84}C_2^3\quad 0-\frac{9}{70}C_1C_2-\frac{41}{210}C_2^2+0+\frac{1}{140}C_2+0$$

$$=-\frac{33}{560}C_1^3+\frac{86}{140}C_1^2C_2+\frac{3517}{1680}C_1C_2^2+\frac{131}{84}C_2^3-\frac{3}{80}C_1^2-\frac{271}{280}C_1C_2-\cdots\cdots$$

<u>Too Complicated</u>

$$\gamma^4\left\{\frac{1}{35}C_1^3+\frac{33}{280}C_1^2C_2-\frac{97}{140}C_1C_2^2-\frac{131}{168}C_2^3+\frac{9}{140}C_1^2+\frac{27}{140}C_1C_2+\frac{41}{280}C_2^2-\frac{1}{70}C_1-\frac{3}{280}C_2-\frac{11}{140}\right\}+\left\{\frac{2}{3}C_1+C_2-\frac{2}{3}\right\}+K\gamma^2=0 \tag{12}$$

$$\gamma^4\left\{\frac{1}{420}C_1^3+\frac{487}{840}C_1^2C_2+\frac{13}{15}C_1C_2^2+\frac{8017}{27720}C_2^3+0-\frac{1}{420}C_1C_2-\frac{1}{280}C_2^2-\frac{1}{140}C_1-\frac{163}{280}C_2+\frac{1}{210}\right\}-\frac{1}{15}C_2=0$$

$$\frac{d}{dx}\left(x\frac{dv}{dx}\right) = \frac{v}{x}\left\{1+\Lambda x^2(v+x)(v+2x)\right\} + x^2$$

The bn.dary conditions; $x=0, \quad v=0$

$x=a, \quad v=0$

Starting with the function $v = cx\left[1-\left(\frac{x}{a}\right)^2\right]$ then

$$\frac{d}{dx}\left(x\frac{dv}{dx}\right) = c\left[1-\left(\frac{x}{a}\right)^2\right]\left\{1+\Lambda x^2\left[(c+1)x - cx\left(\frac{x}{a}\right)^2\right]\left[(c+2)x - cx\left(\frac{x}{a}\right)^2\right]\right\} + x^2$$

$$= c\left[1-\left(\frac{x}{a}\right)^2\right] + c\Lambda x^2\left[1-\left(\frac{x}{a}\right)^2\right]\left[(c+1)x - cx\left(\frac{x}{a}\right)^2\right]\left[(c+2)x - cx\left(\frac{x}{a}\right)^2\right] + x^2$$

$$= c\left[1-\left(\frac{x}{a}\right)^2\right] + c\Lambda\left\{(c+1)(c+2)x^4 - (2c^2+5c+4)x^4\left(\frac{x}{a}\right)^2 + 2c(c+2)x^4\left(\frac{x}{a}\right)^4 - c^2x^4\left(\frac{x}{a}\right)^6\right\} + x^2$$

$$\frac{dv}{dx} = c\left[1-\frac{1}{3}\left(\frac{x}{a}\right)^2\right] + c\Lambda\left\{\frac{(c+1)(c+2)}{5}x^4 - \frac{(2c^2+5c+4)}{7}x^4\left(\frac{x}{a}\right)^2 + \frac{2}{9}c(c+2)x^4\left(\frac{x}{a}\right)^4 - \frac{c^2}{11}x^4\left(\frac{x}{a}\right)^6\right\} + \frac{1}{3}x^2$$

$$v = cx\left[1-\frac{1}{9}\left(\frac{x}{a}\right)^2\right] + c\Lambda\left\{\frac{(c+1)(c+2)}{25} - \frac{(2c^2+5c+4)}{49}\left(\frac{x}{a}\right)^2 + \frac{2}{81}c(c+2)\left(\frac{x}{a}\right)^4 - \frac{c^2}{121}\left(\frac{x}{a}\right)^6\right\}x^5 + \frac{1}{9}x^3$$

$$\therefore \quad 1-\left(\frac{x}{a}\right)^2 = \left[1-\frac{1}{9}\left(\frac{x}{a}\right)^2\right] + \Lambda\left\{\frac{(c+1)(c+2)}{25} - \frac{(2c^2+5c+4)}{49}\left(\frac{x}{a}\right)^2 + \frac{2}{81}c(c+2)\left(\frac{x}{a}\right)^4 - \frac{c^2}{121}\left(\frac{x}{a}\right)^6\right\}x^4 + \frac{1}{9}\frac{x^2}{c}$$

$$-\frac{3}{4} = \left[1 - \frac{1}{36}\right] + \Lambda\left\{\frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{1}{81\times 8}\, C(C+2)\right.$$

$$\left. - \frac{C^2}{121}\frac{1}{64}\right\}\frac{a^4}{16} + \frac{1}{36}\frac{a^2}{C}$$

$$\left\{\frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{8\times 81} - \frac{C^2}{64\times 121}\right\}\Lambda$$

$$= -\frac{2}{9}\frac{16}{a^4} - \frac{1}{36}\frac{1}{C}\frac{16}{a^2} = -\frac{32}{9}\frac{1}{a^4} - \frac{4}{9}\frac{1}{C}\frac{1}{a^2}$$

$$= \frac{4}{9}\left[-8\frac{1}{a^4} - \frac{1}{C}\frac{1}{a^2}\right]$$

$$\frac{\partial\Lambda}{\partial a^2} = 0, \qquad \frac{16}{a^6} + \frac{1}{C}\frac{1}{a^4} = 0$$

$$\frac{16}{a^2} + \frac{1}{C} = 0 \qquad -16C = a^2$$

Ито

$$\left\{\frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{8\times 81} - \frac{C^2}{64\times 121}\right\}\Lambda$$

$$= \frac{4}{9}\left[-\frac{8}{16^2}\frac{1}{C^2} + \frac{1}{C^2}\frac{1}{16}\right]$$

$$= \frac{1}{36}\frac{1}{C^2}\frac{1}{2} = \frac{1}{72}\frac{1}{C^2}$$

$$\left[\frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{8\times 81} - \frac{C^2}{64\times 121}\right]\frac{1}{\phi^2}$$

$$= \frac{1}{3}\frac{1}{C^2}$$

$$\phi = \sqrt{3}\,C\left[\frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{8\times 81} - \frac{C^2}{64\times 121}\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3}\,C}{5\times 7\times 9\times 11\times 8}\left[49\times 81\times 121\times 64\,(C^2+3C+2)\right.$$

$$- 25\times 81\times 121\times 16\,(2C^2+5C+4) + 25\times 49\times 121\times 8\,(C^2+2C)$$

$$\left.- 25\times 49\times 81\,C^2\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3}\,C}{5\times 7\times 9\times 11\times 8}\left[23981711\,C^2 + 74972408\,C + 45790272\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3\times 23981711}\,C}{5\times 7\times 9\times 11\times 8}\left[C^2 + 3.12644C + 1.90938\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3}\,C}{5}\left[C^2 + 3.12644C + 1.90938\right]^{\frac{1}{2}}$$

$$2\left[C^2+3.12644C+1.90938\right] + C\left[2C+3.12644\right] = 0$$

$$\begin{array}{l} C^2 + 3.12644C + 1.90938 \\ C^2 + 1.56322C + 0 \\ \hline C^2 + 2.34483C + 0.95469 \end{array}$$

$$C = -1.17242 \pm \sqrt{1.37457 - 0.95469}$$

$$= -1.17242 \pm \sqrt{0.41988} = -1.17242 \pm 0.64798$$

$C = \begin{matrix} -0.52444 \\ -1.82040 \end{matrix}$

$c^2 + 3.12644c + 1.90938 = -\frac{c}{2}\left[2c + 3.12644\right]$

$= -C\left[C + 1.56322\right] = + 0.46817$

Too big

The differential equation

$$\alpha^2 \frac{d^2\Theta}{d\alpha^2} + \alpha \frac{d\Theta}{d\alpha} - \Theta = C_1 \alpha^3 + C_2 \alpha^2 \Theta (\Theta^2 - \alpha^2)$$

Put $\alpha = \lambda x$

$\Theta = \lambda y$

$$\lambda \left[x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y \right] = C_1 \lambda^3 x^3 + C_2 \lambda^5 x^2 y (y^2 - x^2)$$

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = C_1 \lambda^2 \cdot x^3 + C_2 \lambda^4 \; x^2 y (y^2 - x^2)$$

Let $C_1 \lambda^2 = 1$, $\lambda = \frac{1}{\sqrt{C_1}}$

The equation becomes

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x^3 + \left(\frac{C_2}{C_1^2} \right) x^2 y (y^2 - x^2)$$

$$\frac{C_2}{C_1^2} = \frac{1}{K_2} \cdot \frac{K_2^2}{K_1^2} = \frac{K_2}{K_1^2} = \frac{1}{6\beta^4} \left(\frac{t}{r}\right)^2 \frac{E^2 \beta^4}{4\sigma^2} = \frac{1}{24} \frac{1}{\left[\frac{\sigma}{E(\frac{t}{r})}\right]^2}$$

$$= \frac{1}{24} \frac{1}{\phi^2} \qquad \text{where} \qquad \phi = \frac{\sigma}{E(\frac{t}{r})}$$

$$\alpha = \frac{\alpha_r}{\beta}, \qquad x = \frac{\alpha}{\lambda} = \frac{\alpha_r}{\beta} \cdot \sqrt{C_1} = \frac{\alpha_r}{\beta} \sqrt{\frac{K_1}{K_2}}$$

$$= \frac{\alpha_r}{\beta} \cdot \sqrt{\frac{2\sigma}{E\beta^2} \cdot \frac{6\beta^4}{(\frac{t}{r})^2}} = \frac{2\alpha_r}{(\frac{t}{r})} \sqrt{\frac{3\sigma}{E}}$$

$$= \frac{2\sqrt{3}\,\alpha_r}{(\frac{t}{r})^{\frac{1}{2}}} \phi$$

$$y = \frac{2\sqrt{3}\,\Theta_r}{(\frac{t}{r})^{\frac{1}{2}}} \phi$$

α_r, Θ_r, physical variables

$$x = \frac{1}{\sqrt{2\Lambda \frac{t}{r}}}\,\alpha_r, \quad y = \frac{1}{\sqrt{2\Lambda \frac{t}{r}}}\,\Theta_r$$

where $\Lambda = \frac{1}{24\phi^2}$.

$$\boxed{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y\left\{1 + \Lambda x^2(y^2 - x^2)\right\} = x^3}$$

For corresponding variation problem we write the differential equation as

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{y}{x} - \Lambda x y (y^2 - x^2) - x^2 = 0.$$

$$\boxed{I = \frac{\Lambda}{2}\int_0^x (y^2 - x^2)^2 x\,dx + \int_0^x \left[\left(\frac{dy}{dx} - 1\right)^2 + \left(\frac{y}{x} - 1\right)^2\right] x\,dx + 2\int_0^x x^2 y\,dx}$$

To transform the differential equation into integral equation

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = \left\{\frac{y}{x} + \Lambda x y (y^2 - x^2)\right\} + x^2$$

$$x\frac{dy}{dx} = \int_0^x \left\{\frac{y}{\xi} + \Lambda \xi y (y^2 - \xi^2)\right\} d\xi + \frac{x^3}{3} + C_1$$

When $\frac{dy}{dx} = S_0$ at $x = 0$, $\therefore C_1 = 0$

$$\frac{dy}{dx} = \frac{1}{x}\int_0^x \left\{\frac{y}{\xi} + \Lambda \xi y (y^2 - \xi^2)\right\} d\xi + \frac{x^2}{3} + \frac{C_1}{x}$$

$$y = \int_0^x \frac{dt}{t}\int_0^t \left\{\frac{y}{\xi} + \Lambda \xi y (y^2 - \xi^2)\right\} d\xi + \frac{x^3}{9} + C_1 \log x + C_2$$

When $x = x_0$, $y = x_0$; $x = 0$, $y = 0$

$$y = \int_0^x \log\left(\frac{x}{t}\right)\left\{\frac{y}{t} + \Delta t y (y^2 - t^2)\right\} dt + \frac{x^3}{9} + C_1 \log x + C_2$$

$$x_e = \int_0^{x_e} \log\left(\frac{x_e}{t}\right)\left\{\frac{y}{t} + \Delta t y (y^2 - t^2)\right\} dt + \frac{x_e^3}{9} + C_1 \log x_e + C_2$$

$$0 = C_1(-\infty) + C_2$$

$$\therefore \quad C_1 = 0, \quad C_2 = 0.$$

$$\therefore \quad \boxed{y = \int_0^x \log\left(\frac{x}{t}\right)\left\{\frac{y}{t} + \Delta t y (y^2 - t^2)\right\} dt + \frac{x^3}{9}}$$

this is the integral equation for "y" of second kind.

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the total energy

$$\frac{E\left(\frac{t}{r}\right)}{4}\beta^6 \int_0^1 (\Theta^2 - \alpha^2)^2 \alpha\, d\alpha + \frac{E\left(\frac{t}{r}\right)^3}{12}\beta^2 \int_0^1 \left\{\left(\frac{d\Theta}{d\alpha} - 1\right)^2 + \left(\frac{\Theta}{\alpha} - 1\right)^2\right\} \alpha\, d\alpha$$

$$+ p\beta^4 \int_0^1 \alpha^2 \Theta\, d\alpha$$

$$\frac{1}{4}\beta^2 \int_0^1 (\Theta^2 - \alpha^2)^2 \alpha\, d\alpha + \frac{\left(\frac{t}{r}\right)^2}{12}\frac{1}{\beta^2}\int_0^1 \left\{\left(\frac{d\Theta}{d\alpha} - 1\right)^2 + \left(\frac{\Theta}{\alpha} - 1\right)^2\right\}\alpha\, d\alpha + \left(\frac{2 \sigma}{E}\right)\int_0^1 \alpha^2 \Theta\, d\alpha$$

$$\beta^2 = \frac{\left(\frac{t}{r}\right)}{\sqrt{3}}\left\{\frac{\int_0^1 \left\{\left(\frac{d\Theta}{d\alpha} - 1\right)^2 + \left(\frac{\Theta}{\alpha} - 1\right)^2\right\}\alpha\, d\alpha}{\int_0^1 (\Theta^2 - \alpha^2)^2 \alpha\, d\alpha}\right\}^{\frac{1}{2}}$$

Put $y = w + x$ $\qquad y = w + x \qquad y^2 = w^2 + 2wx + x^2$

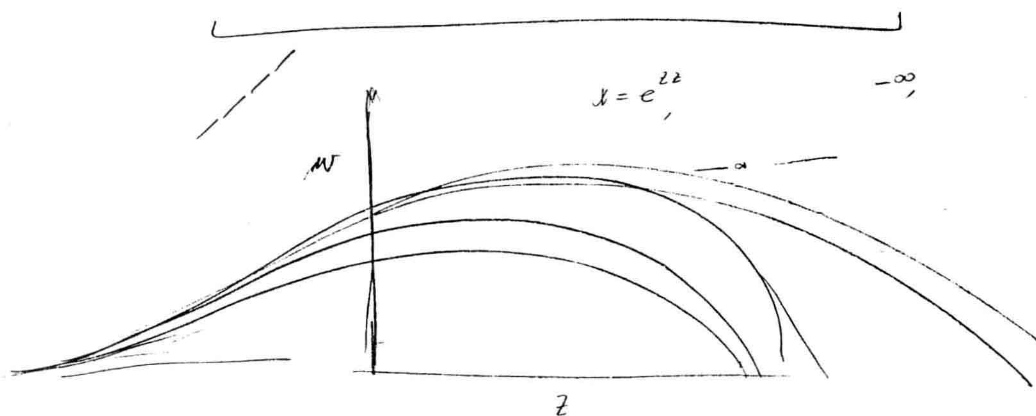
$$\frac{dy}{dx} = \frac{dw}{dx} + 1, \qquad \frac{d^2y}{dx^2} = \frac{d^2w}{dx^2}$$

$$x^2\frac{d^2w}{dx^2} + x\frac{dw}{dx} + x - (w+x)\left\{1 + \Lambda x^2(w^2 + 2wx)\right\} = x^3$$

$$x^2\frac{d^2w}{dx^2} + x\frac{dw}{dx} - w\left\{1 + \Lambda x^2(w+2x)(w+x)\right\} = x^3$$

$$\frac{d^2w}{dz^2} - w\left\{1 + \Lambda e^{2z}(w + 2e^z)(w + e^z)\right\} = e^{3z}$$

$x = e^z$, $\quad -\infty$,



2(3)
a

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y\left\{1+\Delta x^2(y^2-x^2)\right\}=x^3$$

Put $\quad v=y-x \qquad y=v+x$

$$x^2\frac{d^2v}{dx^2}+x\frac{dv}{dx}-v\left\{1+\Delta x^2(v+x)(v+2x)\right\}=x^3$$

Put $\quad w=x^{\frac{1}{2}}v\ , \quad v=\frac{w}{x^{\frac{1}{2}}}$

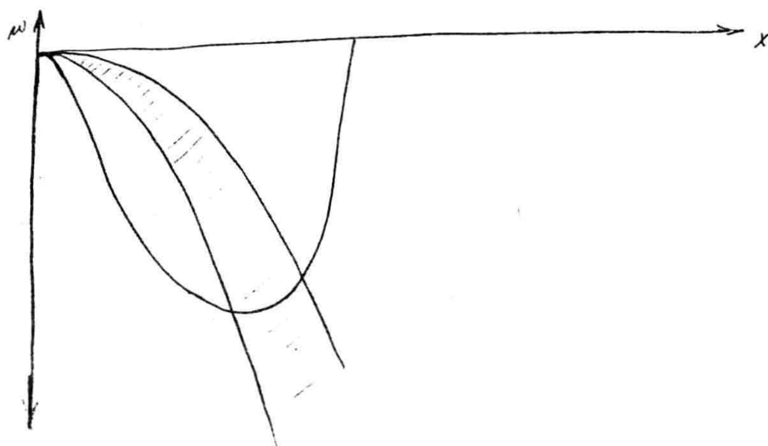
$$\frac{dv}{dx}=\frac{1}{x^{\frac{1}{2}}}\frac{dw}{dx}-\frac{1}{2}\frac{1}{x^{3/2}}w$$

$$\frac{d^2v}{dx^2}=\frac{1}{x^{\frac{1}{2}}}\frac{d^2w}{dx^2}-\frac{1}{x^{3/2}}\frac{dw}{dx}+\frac{3}{4}\frac{1}{x^{5/2}}w$$

$$\therefore \quad x^{3/2}\frac{d^2w}{dx^2}-x^{\frac{1}{2}}\frac{dw}{dx}+\frac{3}{4}\frac{1}{x^{\frac{1}{2}}}w+x^{\frac{1}{2}}\frac{dw}{dx}-\frac{1}{2}\frac{1}{x^{1/2}}w$$

$$-\frac{w}{x^{\frac{1}{2}}}-\Delta\frac{w}{x^{\frac{1}{2}}}x^2\left(\frac{w}{x^{\frac{1}{2}}}+x\right)\left(\frac{w}{x^{\frac{1}{2}}}+2x\right)=x^3$$

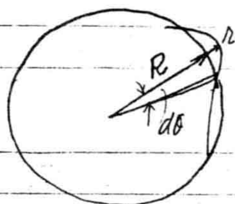
$$\frac{d^2w}{dx^2}=\frac{3}{4}\frac{w}{x^2}+\Delta\frac{w}{x}\left(w+x^{3/2}\right)\left(w+2x^{3/2}\right)+x^{3/2}$$



$$r = R - w$$

$$\psi = \theta + \frac{u}{R}$$

$$(ds)_0^2 = R^2 d\theta^2$$



$$(ds)^2 = (dr)^2 + r^2 d\psi^2$$

$$= d\theta^2\left(\frac{dw}{d\theta}\right)^2 + (R-w)^2\left[1 + \frac{1}{R}\frac{du}{d\theta}\right]^2 d\theta^2$$

$$\varepsilon = \frac{d\theta\left[\sqrt{\left(\frac{dw}{d\theta}\right)^2 + (R-w)^2\left[1+\frac{1}{R}\frac{du}{d\theta}\right]^2} - R\right]}{R\, d\theta}$$

$$= \sqrt{\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 + \left(1 + \frac{w}{R}\right)^2\left(1 + \frac{1}{R}\frac{du}{d\theta}\right)^2} - 1$$

$$= \sqrt{1 - 2\left(\frac{w}{R}\right) + 2\left(\frac{1}{R}\frac{du}{d\theta}\right) + \left(\frac{1}{R}\frac{dw}{d\theta}\right)^2} - 1$$

$$= -\left(\frac{w}{R}\right) + \frac{1}{R}\frac{du}{d\theta} + \frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 = \frac{1}{R}\frac{du}{d\theta} - \frac{w}{R} + \frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 = \varepsilon$$

The total strain energy

$$E_s = \frac{EI}{2}\int\left(\frac{d^2w}{d\theta^2} + \frac{du}{d\theta}\right)^2 \frac{1}{R^4} R\, d\theta + \frac{EA}{2}\int\left[\frac{1}{R}\frac{du}{d\theta} - \frac{w}{R} + \frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2\right]^2 R\, d\theta$$

where I moment of inertia + A area

$$E_s = \frac{E}{2}\left\{\frac{I}{R}\int\left(\frac{1}{R}\frac{d^2w}{d\theta^2} + \frac{1}{R}\frac{du}{d\theta}\right)^2 d\theta + AR\int\left[\frac{1}{R}\frac{du}{d\theta} - \frac{w}{R} + \frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2\right]^2 d\theta\right\}$$

$$= \frac{EAR}{2}\left\{\left(\frac{i}{R}\right)^2\int\left(\frac{1}{R}\frac{d^2w}{d\theta^2} + \frac{1}{R}\frac{du}{d\theta}\right)^2 d\theta + \int\left[\frac{1}{R}\frac{du}{d\theta} - \frac{w}{R} + \frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2\right]^2 d\theta\right\}$$

If $u=0$,

$$E_s = \frac{EAR}{2}\left\{\left(\frac{i}{R}\right)^2\int\left[\frac{1}{R}\frac{d^2w}{d\theta^2}\right]^2 d\theta + \int\left[\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 - \frac{w}{R}\right]^2 d\theta\right\}$$

For frame

For stiffener

$$E_s = \frac{EI_s}{2}\int\left(\frac{d^2w}{dx^2}\right)^2 dx$$

For frame: Putting $\frac{w}{R} = \beta\left[\cos(n\theta) + \cos(2n\theta)\right]$

$$\frac{1}{R}\frac{dw}{d\theta} = \beta\left[-n\sin n\theta - 2n\sin(2n\theta)\right]$$

$$\frac{1}{R}\frac{d^2w}{d\theta^2} = \beta\left[-n^2\cos n\theta - 4n^2\cos(2n\theta)\right]$$

$$= -\beta n^2\left[\cos n\theta + 4\cos(2n\theta)\right]$$

$$2\int_0^{\pi/n}\left[\frac{1}{R}\frac{d^2w}{d\theta^2}\right]^2 d\theta = 2\beta^2 n^3\int_0^{\pi}(\cos\psi + 4\cos 2\psi)^2 d\psi$$

$$= \pi\beta^2 n^3\cdot 17 = 17\pi\beta^2 n^3$$

$$\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 = \beta^2 n^2\left[\sin n\theta + 2\sin(2n\theta)\right]^2$$

$$= \beta^2 n^2\left[\sin^2 n\theta + 4\sin n\theta\sin(2n\theta) + 4\sin^2(2n\theta)\right]$$

$$\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 = \beta^2 n^2\left[\frac{1}{2}(1-\cos 2n\theta) + 2(\cos n\theta - \cos 3n\theta) + 2(1-\cos 4n\theta)\right]$$

$$= \beta^2 n^2\left[\frac{5}{2} + 2\cos n\theta - \frac{1}{2}\cos 2n\theta - 2\cos 3n\theta - 2\cos 4n\theta\right]$$

$$\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 - \frac{w}{R} = \beta^2 n^2\left[\frac{5}{4} + \cos n\theta - \frac{1}{4}\cos 2n\theta - \cos 3n\theta - \cos 4n\theta\right]$$

$$- \beta\left[\cos n\theta + \cos 2n\theta\right]$$

$$= \beta\left[\frac{5}{4}\beta n^2 + (\beta n^2 - 1)\cos n\theta - \left(\frac{\beta n^2}{4} + 1\right)\cos 2n\theta - \beta n^2 \cos 3n\theta - \beta n^2 \cos 4n\theta\right]$$

$$2\int_0^{\frac{\pi}{n}}\left[\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2 - \frac{w}{R}\right]^2 d\theta$$

$$= \frac{\pi\beta^2}{n}\left[\frac{25}{8}(\beta n^2)^2 + (\beta n^2 - 1)^2 + \left(\frac{\beta n^2}{4} + 1\right)^2 + 2(\beta n^2)^2\right]$$

$$= \frac{\pi\beta^2}{n}\left[\frac{99}{16}(\beta n^2)^2 - \frac{3}{2}(\beta n^2) + 2\right]$$

$$E_s = \frac{EAR}{2}\left\{\left(\frac{i_f}{R}\right)^2 17\pi\beta^2 n^3 + \frac{\pi\beta^2}{n}\left[\frac{99}{16}(\beta n^2)^2 - \frac{3}{2}(\beta n^2) + 2\right]\right\}$$

$$= \frac{EA_f R\pi\beta^2}{2}\left\{\left(\frac{i_f}{R}\right)^2 17n^3 + \frac{1}{n}\left[\frac{99}{16}(\beta n^2)^2 - \frac{3}{2}(\beta n^2) + 2\right]\right\}$$

$$= \frac{EA_f R\pi}{2}\left\{\beta^2\left[\left(\frac{i_f}{R}\right)^2 17n^3 + \frac{2}{n}\right] + \beta^4\frac{99}{16}n^3 - \beta^3\frac{3}{2}n\right\}$$

$$\boxed{\mathcal{E}_{st_f} = \frac{EA_f R\pi}{2}\left\{\frac{1}{R^2}N_1 a^2\left[\left(\frac{i_f}{R}\right)^2 17n^3 + \frac{2}{n}\right] + \frac{1}{R^4}N_3 a^4\frac{99}{16}n^3 - \frac{1}{R^3}N_2 a^3\frac{3}{2}n\right\}}$$

$$\mathcal{E}_{tot._s} = 2\pi^5\left(\frac{1}{n}\right)a^2\left(\frac{R}{d}\right)\left(\frac{EI_s}{L^3}\right)$$

$$I_{ex.} = \frac{1}{2}\pi^2\left(\frac{P}{d}\right)\left(\frac{R}{L}\right)a^2(M_1 + R_1)$$

Therefore the total potential energy of the system

$$\mathcal{E} = \frac{EA_fR\pi}{2}\left\{\frac{1}{R^2}N_1a^2\left[\left(\frac{i_f}{R}\right)^2 17n^3 + \frac{2}{n}\right] - \frac{1}{R^3}N_2a^3\frac{3}{2}n + \frac{1}{R^4}N_3a^4\frac{99}{16}n^3\right\}$$

$$+ 2\pi^5\left(\frac{1}{n}\right)a^2\left(\frac{R}{d}\right)\frac{EI_s}{L^3} - \frac{1}{2}\pi^2\left(\frac{P}{d}\right)\left(\frac{R}{L}\right)a^2(M_1+R_1)$$

$$\frac{\partial\mathcal{E}}{\partial a} = 0 \qquad \frac{\partial\mathcal{E}}{\partial n} = 0$$

the first condition gives

$$EA_fR\pi\left\{\frac{N_1}{R^2}\left[\left(\frac{i_f}{R}\right)^2 17n^3 + \frac{2}{n}\right] - \frac{1.5}{R^3}N_2an + \frac{N_3}{R^4}a^2\frac{99}{8}n^3\right\}$$

$$+ 4\pi^5\left(\frac{1}{n}\right)\left(\frac{R}{d}\right)\frac{EI_s}{L^3} = \pi^2\left(\frac{P}{d}\right)\left(\frac{R}{L}\right)(M_1+R_1)$$

$$\frac{(-1)(-2)}{2!}$$

Assuming $n=1$

$$EA_f R \left\{ \frac{N_1}{R^2}\left[17\left(\frac{i_f}{R}\right)^2+2\right] - 2.25\frac{N_2}{R^3}a + 12.375\frac{N_3}{R^4}a^2 \right\}$$

$$+ 4\pi^4\left(\frac{R}{d}\right)\frac{EI_s}{L^3} = \pi\left(\frac{P}{d}\right)\left(\frac{R}{L}\right)(M_1+R_1)$$

or $\qquad 2.25\, N_2 = 24.75\, N_3/R \cdot a$

$$\therefore \quad EA_f \frac{1}{R}\left\{ N_1\left[17\left(\frac{i_f}{R}\right)^2+2\right] - 0.1022\frac{N_2^2}{N_3}\right\} + 4\pi^4\left(\frac{R}{d}\right)\frac{EI_s}{L^3}$$

$$= \pi\left(\frac{P}{d}\right)\left(\frac{R}{L}\right)(M_1+R_1)$$

$$M_1 = \sin\left(\frac{\pi}{n}\right)\left[1 - \frac{1}{1-n^2} + \frac{\frac{1}{2}}{1-4n^2} - \frac{1}{1-9n^2} + \frac{\frac{1}{2}}{1-16n^2}\right]$$

$$= \sin\left(\frac{\pi}{n}\right)\left(-\frac{1}{1-n^2}\right) \qquad \text{when } n=1$$

$$= -\frac{\sin\frac{\pi}{n}}{1-n^2} = -\frac{\cos\frac{\pi}{n}\cdot\left(-\frac{\pi}{n^2}\right)}{-2n} = \frac{\pi}{2}$$

$$R_1 = \frac{\pi}{4}$$

$$M_1 + R_1 = \frac{3\pi}{4}$$

$$\frac{E}{R}\left\{N_1\left[17\frac{I_e}{R^2}+2A_f\right]-0.1022\frac{N_2^2}{N_3}A_f\right\}+E\;\pi^4\left(\frac{R}{d}\right)\frac{I_s}{L^3}$$

$$=\pi^2\frac{3}{4}\left(\frac{P}{d}\right)\left(\frac{R}{L}\right)$$

$$4N_1=\sum_{k=0}^{n}\left[1-\cos\left(\frac{2\pi k}{n}\right)\right]^2$$

$$=(n+1)-2\sum_{k=0}^{n}\cos\left(\frac{2\pi k}{n}\right)+\sum_{k=0}^{n}\cos^2\left(\frac{2\pi k}{n}\right)$$

$$=(n+1)-2\sum_{k=0}^{n}\cos\left(\frac{2\pi k}{n}\right)+\frac{1}{2}\sum_{k=0}^{n}\left(1+\cos\frac{4\pi k}{n}\right)$$

$$=\frac{3(n+1)}{2}-2\sum_{k=0}^{n}\cos\left(\frac{2\pi k}{n}\right)+\frac{1}{2}\sum_{k=0}^{n}\cos\frac{4\pi k}{n}$$

$$=\frac{3n}{2}-2\sum_{k=0}^{n}\cos k\theta+\frac{1}{2}\sum_{k=0}^{n}\cos k(2\theta) \qquad \theta=\frac{2\pi}{n}$$

$$2\sum_{k=0}^{n}\cos k\theta=\sum_{k=0}^{n}e^{ik\theta}+\sum_{k=0}^{n}e^{-ik\theta}=\sum_{k=0}^{n}(e^{i\theta})^k+\sum_{k=0}^{n}(e^{-i\theta})^k$$

$$(e^{i\theta})\sum_{k=0}^{n}(e^{i\theta})^k-\sum_{k=0}^{n}(e^{i\theta})^k=(e^{i\theta})^{n+1}-1$$

$$\therefore\ \sum_{k=0}^{n}(e^{i\theta})^k=\frac{(e^{i\theta})^{n+1}-1}{e^{i\theta}-1}$$

$$\sum_{k=0}^{n}(e^{i\theta})^k+\sum_{k=0}^{n}(e^{-i\theta})^k=\frac{(e^{i\theta})^{n+1}-1}{e^{i\theta}-1}+\frac{(e^{-i\theta})^{n+1}-1}{e^{-i\theta}-1}$$

$$=\frac{1-\cos\theta+\cos n\theta-\cos(n+1)\theta}{1-\cos\theta}$$

$$=1+\frac{\cos n\theta-\cos(n+1)\theta}{1-\cos\theta}=1+\frac{1-\cos\frac{2\pi}{n}}{1-\cos\frac{2\pi}{n}}$$

$$=2$$

$$\therefore\quad 4N_1=\frac{3(n+1)}{2}-2+\frac{1}{2}=\frac{3(n+1)}{2}-\frac{3}{2}=\frac{3}{2}\left[(n+1)-1\right]$$

$$N_1=\frac{3}{8}n\qquad \text{for}\quad n>2$$

$$8N_2=\sum_{k=0}^{n}\left[1-\cos\left(\frac{2\pi k}{n}\right)\right]^3$$

$$=(n+1)-3\sum_{k=0}^{n}\cos\frac{2\pi k}{n}+3\sum_{k=0}^{n}\cos^2\frac{2\pi k}{n}-\sum_{k=0}^{n}\cos^3\frac{2\pi k}{n}$$

$$=\frac{5}{2}(n+1)-\frac{15}{2}+3-\frac{1}{2}=\frac{5}{2}\left[(n+1)-2\right]=\frac{5}{2}(n-1)$$

$$N_2=\frac{5}{16}n\qquad \text{for}\quad n>2$$

$$16N_3=\sum_{k=0}^{n}\left[1-\cos\left(\frac{2\pi k}{n}\right)\right]^4$$

$$=\sum_{k=0}^{n}\left[1-4\cos\left(\frac{2\pi k}{n}\right)+6\cos^2\left(\frac{2\pi k}{n}\right)-4\cos^3\left(\frac{2\pi k}{n}\right)+\cos^4\left(\frac{2\pi k}{n}\right)\right]$$

$$=\frac{35}{8}(n+1)-14+7-2+\frac{1}{4}=\frac{35}{8}w$$

$$N_3 = \frac{35n}{128}$$

$$\frac{E}{R}\left\{ \frac{3}{8}(m+1)\left[17\frac{I_f}{R^2}+2A_f\right] - 0.1022 \,\frac{\frac{25}{16^2}(m+1)}{\frac{35}{16\times 8}} A_f \right\}$$

$$\therefore \frac{3\pi^2}{4}\left(\frac{P}{d}\right)\left(\frac{R}{L}\right) = E\left[\frac{1}{R}\left\{\frac{3}{8}(m+1)\left(17\frac{I_f}{R^2}+2A_f\right) - 0.1022\times\frac{5}{14}(m+1)A_f\right\}\right.$$

$$\left. + \pi^4\left(\frac{R}{d}\right)\frac{I_s}{L^3}\right]$$

$$\text{moment} = 4\int_0^{\frac{\pi}{2}} dP\cdot R\cos\varphi = 4R^2\left(\frac{P}{d}\right)\int_0^{\frac{\pi}{2}}\cos^2\varphi\, d\varphi$$

$$= \pi R^2\left(\frac{P}{d}\right) = \mathcal{M}$$

$$\frac{3}{4}\pi\left(\frac{\mathcal{M}}{R^2}\right)\left(\frac{R}{L}\right) = E(m+1)\,\frac{1}{R}\left\{\frac{3}{8}\left(17\frac{I_f}{R^2}+2A_f\right) - 0.1022\times\frac{5}{14}A_f\right\}$$

$$+ E\pi^4\left(\frac{R}{d}\right)\frac{I_s}{L^3}$$

$$M = \frac{4}{3\pi}ERL\left[\frac{(m+1)}{R}\left\{\frac{3}{8}\left(17\frac{I_f}{R^2}+2A_f\right) - 0.1022\times\frac{5}{14}A_f\right\}\right.$$

$$\left. + \pi^4\left(\frac{R}{d}\right)\frac{I_s}{L^3}\right]$$

$m = 31$, $R = 15.76''$, $L = 64''$, $A_f = 0.0291$

$d = 2.53''$, $I_f = 0.0000\,1537$

$I_s = 0.000\,374$

$$\frac{m+1}{R}\left\{\frac{3}{8}\left(17\frac{I_f}{R^2}+2A_f\right)-0.1022\times\frac{5}{14}A_f\right\}$$

$$=\frac{32}{15.76}\left\{0.375\left(17\times\frac{0.00001537}{15.76^2}+2\times0.0291\right)-0.1022\times\frac{5}{14}\times0.0291\right\}$$

$$=\frac{32}{15.76}\left\{0.375\left(0.000001051+0.0582\right)-0.001062\right\}$$

$$=\frac{32}{15.76}\times0.02074=0.0421$$

$$\pi^4\left(\frac{15.76}{2.53}\right)\frac{0.000374}{64^3}=\frac{0.0000554}{64}$$

$$M=\frac{1.3333}{\pi}\times10.6\times10^6\times15.76\times64\times0.0421$$ Too big

Approximate Calculation

Assuming equipartition of energy in frame

Putting $\left(\frac{i}{R}\right)\left[\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{1}{R}\frac{du}{d\theta}\right]=\frac{1}{R}\frac{du}{d\theta}-\frac{w}{R}+\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2$

$$\left(1-\frac{i}{R}\right)\frac{1}{R}\frac{du}{d\theta}=\frac{i}{R}\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{w}{R}-\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2$$

$$\frac{1}{R}\frac{du}{d\theta}=\frac{1}{1-\left(\frac{i}{R}\right)}\left[\frac{i}{R}\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{w}{R}-\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2\right]$$

$$\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{1}{R}\frac{du}{d\theta}=\frac{1}{1-\left(\frac{i}{R}\right)}\left\{\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{w}{R}-\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2\right\}$$

$$\cong\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{w}{R}-\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2$$

For the assumed buckling form

$$\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{w}{R}-\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2$$

$$=-\beta\left\{\left[n^2\cos n\theta+4n^2\cos(2n\theta)\right]+\frac{5}{4}\beta n^2+(\beta n^2-1)\cos n\theta\right.$$
$$\left.-\left(\frac{\beta n^2}{4}+1\right)\cos 2n\theta-\beta n^2\cos 3n\theta-\beta n^2\cos 4n\theta\right\}$$

$$=-\beta\left\{\frac{5}{4}\beta n^2+(\beta n^2+n^2-1)\cos n\theta+\left(4n^2-\frac{\beta n^2}{4}-1\right)\cos 2n\theta\right.$$
$$\left.-\beta n^2\cos 3n\theta-\beta n^2\cos 4n\theta\right\}$$

$$\int\left[\frac{1}{R}\frac{d^2w}{d\theta^2}+\frac{w}{R}-\frac{1}{2}\left(\frac{1}{R}\frac{dw}{d\theta}\right)^2\right]^2d\theta$$

$$=\frac{\pi\beta^2}{n}\left\{\frac{25}{8}(\beta n^2)^2+(\beta n^2+n^2-1)^2+\left(4n^2-\frac{\beta n^2}{4}-1\right)^2+2(\beta n^2)^2\right\}$$

$$= \frac{\pi\beta^2}{n}\left\{ \frac{25}{8}(\beta n^2)^2 + (\beta n^2)^2 + n^4 + 1 + 2\beta n^4 - 2\beta n^2 - 2n^2 \right.$$

$$\left. + 16n^4 - 2\beta n^4 - 8n^2 + \frac{\beta n^2}{2} + 1 + 2(\beta n^2)^2 + \frac{(\beta n^2)^2}{16} \right\}$$

$$= \frac{\pi\beta^2}{n}\left\{ \frac{99}{16}(\beta n^2)^2 - \frac{3}{2}\beta n^2 + 2 + 17n^4 - 10n^2 \right\}$$

$$= \frac{\pi}{n}\left\{ \frac{99}{16} n^4 \beta^4 - \frac{3}{2} n^2 \beta^3 + (2 + 17n^4 - 10n^2)\beta^2 \right\}$$

$$\mathcal{E}_{tot_f} = \frac{EI_f}{2R^3}\,\frac{\pi}{n}\left\{ \frac{99}{16} n^4 N_3 \frac{a^4}{R^2} - 1.5 n^2 N_2 \frac{a^3}{R} + (2 + 17n^4 - 10n^2) N_1 a^2 \right\}$$

When $n = 1$

$$\mathcal{E}_{tot_f} = \frac{EI_f}{2R^3}\,\pi\left\{ \frac{99}{16} N_3 \frac{a^4}{R^2} - 1.5 N_2 \frac{a^3}{R} + 9 N_1 a^2 \right\}$$

$$\boxed{\frac{EI_f}{R^3}\left\{ \frac{99}{8} N_3 \frac{a^2}{R^2} - 2.25 N_2 \frac{a}{R} + 9N_1 \right\} + 4\pi^4 \left(\frac{R}{d}\right) \frac{EI_s}{L^3} = \pi \left(\frac{P}{d}\right)\left(\frac{R}{L}\right)(M_1 + R_1)}$$

$$\frac{99}{4} N_3 \frac{a}{R} = 2.25 N_2 \quad \Big/ \quad a = \frac{1}{11}\frac{N_2}{N_3} R$$

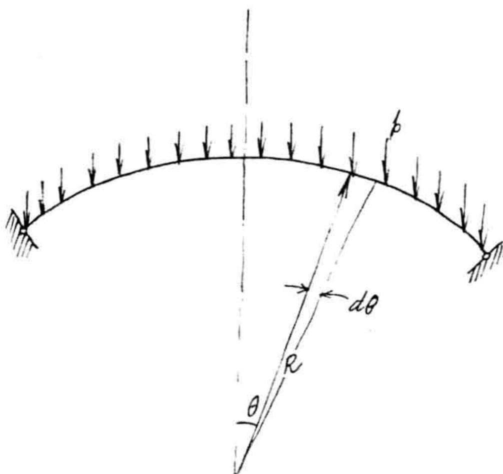
$$\frac{EI_f}{R^3}\left\{ 9N_1 - \frac{9}{88}\frac{N_2^2}{N_3} \right\} + 4\pi^4\left(\frac{R}{a}\right)\frac{EI_s}{L^3} = \pi\left(\frac{P}{d}\right)\left(\frac{R}{L}\right)(M_1 + R_1)$$

Approximately, $\frac{3\pi^2}{4}\left(\frac{P}{d}\right)\left(\frac{R}{L}\right) = E\left[\frac{I_f}{R^3}(m+1)\left(\frac{27}{8} - \frac{9}{88}\frac{5}{14}\right) + 4\pi^4\left(\frac{R}{d}\right)\frac{I_s}{L^3}\right]$

$$M = \frac{4}{3\pi} E R L \left\{ (m+1)\frac{I_f}{R^3}\, 3.347 + 4\pi^4 \frac{R}{d}\frac{I_s}{L^3} \right\}$$

$$(m+1)\frac{I_f}{R^3}\, 3.347 = \frac{3.347\ \dfrac{0.00001537}{15.76^2} \times 32}{R}$$

$$= \frac{0.0000043}{R} \qquad \text{Too small}$$



The original form of the shell is spherical.

Now suppose the deflected form of the shell is axially symmetrical.

$$\theta_1 = \theta_0 + \theta_0 f(\theta_0) = \theta_0[1+f(\theta_0)]$$

$$R = R_0 + R\, g(\theta_0)$$

$$= R[1+g(\theta_0)]$$

The original length of the element $(dS)_0 = R(d\theta)_0$

The new length of the element

$$= \sqrt{R^2(d\theta_1)^2 + (dR_1)^2}$$

$$= \sqrt{R^2[1+g(\theta_0)]^2\left[\{1+f(\theta_0)\}d\theta_0 + \theta_0 f'(\theta_0)\, d\theta_0\right]^2 + R^2[g'(\theta_0)]^2(d\theta_0)^2}$$

$$= R\sqrt{[1+g(\theta_0)]^2[1+f(\theta_0)+\theta_0 f'(\theta_0)]^2 + [g'(\theta_0)]^2}\; d\theta_0$$

If the deflection is inextensional in the sense that

$(ds)_0 = (ds)_1$

Then $$[1+g(\theta_0)]^2[1+f(\theta_0)+\theta_0 f'(\theta_0)]^2 + [g'(\theta_0)]^2 = 1$$

The distance of the element from the axis is

$$R \cdot \sin\theta_0$$

$\left(\frac{1}{2}E\varepsilon^2 v\right.$

before deflection.

The distance is $r \sin\theta$, after deflection.

$$R[1+g(\theta_0)] \sin[\theta_0(1+f(\theta_0))]$$

$\sin[\theta_0 + \theta_0 f(\theta_0)]$

The change in length of the ring ds is

$$2\pi R\left[[1+g(\theta_0)] \sin\{\theta_0(1+f(\theta_0)\} - \sin\theta_0\right]$$

The strain energy stored in this ds is

$$\frac{1}{2}[E\,\varepsilon]\, t\, ds \cdot 2\pi R\left[[1+g(\theta_0)]\sin[\theta_0[1+f(\theta_0)]] - \sin\theta_0\right] \quad t = \text{thickness}$$

now
$$\varepsilon = [1+g(\theta_0)]\frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1$$

~~$= [1+g(\theta_0)]\{\cos\{\theta_0 f(\theta_0)\} + \cot\theta_0 \sin$~~

The total strain energy

$$= \frac{1}{2}tER^2 2\pi\int_0^{\alpha}\left\{[1+g(\theta_0)]\frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1\right\}^2 (\sin\theta_0)\, d\theta_0$$

Potential energy of the pressure force.

pV where V = volume under the shell

The volume under the shell

$$= \int_0^{\alpha} \frac{1}{3} \; 2\pi R \sin\theta_1 \cdot r \cdot r \, d\theta_1$$

$$= \frac{2\pi R^3}{3} \int_0^{\alpha} [1+g(\theta_0)]^3 \sin\{\theta_0[1+f(\theta_0)]\} \{[1+f(\theta_0)] + \theta_0 f'(\theta_0)\} \, d\theta_0$$

The integral to be minimized is

$$\{\tfrac{t}{R}E\} \int_0^{\alpha} \left\{ [1+g(\theta_0)] \frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1 \right\}^2 \sin\theta_0 \, d\theta_0$$

$$- \frac{2p}{3} \int_0^{\alpha} [1+g(\theta_0)]^3 \sin\{\theta_0[1+f(\theta_0)]\} \times [1+f(\theta_0) + \theta_0 f'(\theta_0)] \, d\theta_0 .$$

To simplify the expression, let us put $\theta_0 f(\theta_0) = h(\theta_0)$

$$I = \{\tfrac{t}{R}E\} \int_0^{\alpha} \left\{ [1+g(\theta_0)] \frac{\sin(\theta_0 + h(\theta_0))}{\sin\theta_0} - 1 \right\}^2 \sin\theta_0 \, d\theta_0$$

$$- \frac{2p}{3} \int_0^{\alpha} [1+g(\theta_0)]^3 \sin(\theta_0 + h(\theta_0)) \cdot [1+h'(\theta_0)] \, d\theta_0$$

The inextensional condition is

$$[1+g(\theta_0)]^2 [1+h'(\theta_0)]^2 + [g'(\theta_0)]^2 - 1 = 0$$

~~[illegible]~~

the Euler - Lagrange differential equation is then

$$\frac{t}{R} E \left\{ \cancel{f}\left[[1+g(\theta_0)] \frac{\sin(\theta_0+h(\theta_0))}{\sin\theta_0} - 1 \right] \cancel{\sin\theta_0} \cdot \frac{\sin(\theta_0+h(\theta_0))}{\cancel{\sin\theta_0}} \right\}$$

$$- \frac{\Delta p}{\cancel{3}} \left\{ \cancel{3} \left[1+g(\theta_0)\right]^2 \sin(\theta_0+h(\theta_0)) \cdot \left[1+h'(\theta_0)\right] \right\}$$

$$+\lambda \left\{ \cancel{f}\left[1+g(\theta_0)\right]\left[1+h'(\theta_0)\right]^2 \right\} - \cancel{f} g''(\theta_0) \Big\} = 0.$$

$\frac{d}{d\theta_0}(\lambda g(\theta_0))$

$$\frac{t}{R} E \left\{ 2 \sin\theta_0 \left[[1+g(\theta_0)] \frac{\sin(\theta_0+h(\theta_0))}{\sin\theta_0} - 1 \right] [1+g(\theta_0)] \frac{\cos(\theta_0+h(\theta_0))}{\sin\theta_0} \right\}$$

$$- \frac{\Delta p}{3} \Big\{ \cancel{[1+g(\theta_0)]^3 \cos(\theta_0+h(\theta_0))[1+h'(\theta_0)]}$$

$$- 3 \left[1+g(\theta_0)\right]^2 g'(\theta_0) \sin(\theta_0+h(\theta_0)) - \cancel{[1+g(\theta_0)]^3 \cos(\theta_0+h(\theta_0))[1+h'(\theta_0)]} \Big\}$$

$$\frac{d}{d\theta_0}\lambda \left\{ \frac{d}{d\theta_0}\left\{[1+g(\theta_0)]^2 f [1+h'(\theta_0)]\right\} \right\} = 0.$$

5)

$$\frac{t}{R}E\left\{[1+g(\theta_0)]\frac{\sin(\theta_0+h(\theta_0))}{\sin\theta_0}-1\right\}\sin(\theta_0+h(\theta_0))$$

$$-\frac{p}{r}[1+g(\theta_0)]^2[1+h'(\theta_0)]\sin(\theta_0+h(\theta_0))$$

$$+\lambda\left\{[1+g(\theta_0)][1+h'(\theta_0)]^2-g''(\theta_0)\right\}-g'(\theta_0)\frac{d\lambda}{d\theta_0}=c. \qquad (1)$$

$$\frac{t}{R}E\left\{[1+g(\theta_0)]\frac{\sin(\theta_0+h(\theta_0))}{\sin\theta_0}-1\right\}[1+g(\theta_0)]\cos(\theta_0+h(\theta_0))$$

$-\frac{p}{3}\Big\{$ ~~$[1+g(\theta_0)]^3\cos(\theta_0+h(\theta_0))\;[1+h'(\theta_0)]$~~

$-3[1+g(\theta_0)]^2 g'(\theta_0)\sin(\theta_0+h(\theta_0))-$ ~~$[1+g(\theta_0)]^3[1+h'(\theta_0)]\cos(\theta_0+h(\theta_0))$~~ $\Big\}$

$$-\lambda\left\{2[1+g(\theta_0)]g'(\theta_0)[1+h'(\theta_0)]+[1+g(\theta_0)]^2h''(\theta_0)\right\} \qquad (2)$$

$$-[1+g(\theta_0)]^2[1+h'(\theta_0)]\frac{d\lambda}{d\theta_0}=0$$

$$[1+g(\theta_0)]^2[1+h'(\theta_0)]^2+[g'(\theta_0)]^2-1=0 \qquad (3)$$

Now if we assume that both

$$g(\theta_0) \text{ and } h(\theta_0)$$

are small quantities ~~and so the quadratic & higher order terms~~ But retain terms of the form $g''(\theta_0) \cdot g(\theta_0) \quad g''(\theta_0)\, g'(\theta_0)$

Then $$\frac{\sin[\theta_0 + h(\theta_0)]}{\sin\theta_0} = 1 + \cot\theta_0 \cdot h(\theta_0)$$

[In the following calculation $\theta_0 \approx \theta$]

$$[1+g(\theta)]\frac{\sin[\theta+h(\theta)]}{\sin\theta} - 1$$

$$= [1+g(\theta)][1+h(\theta)\cot\theta] - 1$$

$$\cong g(\theta) + h(\theta)\cot\theta$$

$$\sin(\theta + h(\theta)) = \sin\theta + h(\theta)\cos\theta$$

$$[1+g(\theta)]^2[1+h'(\theta)][\sin\theta + h(\theta)\cos\theta]$$

$$= [1+2g(\theta)][1+h'(\theta)][\sin\theta + h(\theta)\cos\theta]$$

$$= [1+2g(\theta)+h'(\theta)][\sin\theta + h(\theta)\cos\theta)]$$

$$= \sin\theta + h(\theta)\cos\theta + 2g(\theta)\cdot\sin\theta + h'(\theta)\sin\theta.$$

$$[1+g(\theta)][1+2h'(\theta)] - g''(\theta_0)$$

$$= 1 + g(\theta) + 2h'(\theta) - g''(\theta)$$

Then the first differential equation becomes

$$\frac{t}{R}E\left\{\sin\theta\left[g'(\theta)+h(\theta)\cot\theta\right]\right\}_\theta - p\left\{\sin\theta + h(\theta)\cos\theta + 2g(\theta)\sin\theta + h'(\theta)\sin\theta\right\}$$

$$+\lambda\left\{1+g(\theta)+2h'(\theta)-g''(\theta)\right\} - g'(\theta)\frac{d\lambda}{d\theta} = 0.$$

$$\cos(\theta_0+h(\theta)) = \cos\theta - h(\theta)\sin\theta$$

$$[1+g'(\theta)]\cos(\theta+h(\theta)) = \cos\theta - h(\theta)\sin\theta + g(\theta)\cos\theta.$$

$$\frac{t}{R}E\left\{\cos\theta\left[g(\theta)+h(\theta)\cot\theta\right]\right\} + p\sin\theta\cdot g'(\theta)$$

$$-\lambda\left\{2g'(\theta)+h''(\theta)\right\} - \left\{1+2g(\theta)+h'(\theta)\right\}\frac{d\lambda}{d\theta} = 0$$

$$[1+2g(\theta)][1+2h'(\theta)] - 1 = 0$$

$$g(\theta) + h'(\theta) = 0.$$

From the last equation, it is seen that $g'' \approx h'''$ 8)
there $h'''\cdot h$, $h'''h''$, $h'''h'$ not to be neglected.

$$\frac{t}{R}E\left[h(\theta)\cos\theta - h'(\theta)\sin\theta\right] - p\left\{\sin\theta + h(\theta)\cdot\cos\theta - h'(\theta)\sin\theta\right\}$$

$$+ \lambda\left\{1 + h'(\theta) + h'''(\theta)\right\} + h''(\theta)\frac{d\lambda}{d\theta} = 0.$$

$$\frac{t}{R}E\left[h(\theta)\cdot\cos\theta\cdot\cot\theta - h'(\theta)\cos\theta\right] - p\,g''(\theta)\sin\theta$$

$$+ \lambda\left[h''(\theta)\right] - \left\{1 - h'(\theta)\right\}\frac{d\lambda}{d\theta} = 0.$$

This method of derivation unsatisfactory, because we have to use further differentiation to eliminate λ but then some of the neglected terms may become important.

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9)

Timoshenko's differential equations when there is no bending moment will be

$$\frac{dN_x}{d\theta} + (N_x - N_y)\cot\theta + N_y\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) = 0$$

$$N_x + N_y + qa + N_x\left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta}\right) + N_y\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\cot\theta = 0$$

$$N_x = \frac{Et}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2) - \frac{qa}{2}$$

$$= \frac{Et}{1-\nu^2}\left[\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right) + \nu\left(\frac{u\cos\theta}{a\sin\theta} - \frac{w}{a}\right)\right] - \frac{qa}{2}$$

$$N_y = \frac{Et}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1) - \frac{qa}{2}$$

$$= \frac{Et}{1-\nu^2}\left[\left(\frac{u\cos\theta}{a\sin\theta} - \frac{w}{a}\right) + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] - \frac{qa}{2}$$

$$\frac{dN_x}{d\theta} = \frac{Et}{1-\nu^2}\left[\frac{1}{a}\frac{d^2u}{d\theta^2} - \frac{1}{a}\frac{dw}{d\theta} + \nu\left(\frac{1}{a}\frac{\cos\theta}{\sin\theta}\frac{du}{d\theta} - \frac{u}{a}\frac{1}{\sin^2\theta} - \frac{1}{a}\frac{dw}{d\theta}\right)\right]$$

Therefore the differential equation can be written as

$$\frac{Et}{1-\nu^2}\left[\frac{1}{a}\frac{d^2u}{d\theta^2} - (1+\nu)\frac{1}{a}\frac{dw}{d\theta} + \nu\left(\frac{1}{a}\cot\theta\frac{du}{d\theta} - \frac{u}{a}\frac{1}{\sin^2\theta}\right)\right]$$

$$+ \frac{Et}{1-\nu^2}\left[\left(\frac{du}{a\,d\theta} - \frac{u\cos\theta}{a\sin\theta}\right)(1-\nu)\right]\cot\theta$$

$$+ \left\{\frac{Et}{1-\nu^2}\left[\left(\frac{u\cos\theta}{a\sin\theta} - \frac{w}{a}\right) + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] - \frac{qa}{2}\right\}\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) = 0.$$

10)

Let us put $p = \dfrac{q}{\left(\frac{Et}{1-\nu^2}\right)}$, then

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - (1+\nu)\frac{dw}{a\,d\theta} + \nu\left(\frac{1}{a}\cot\theta\frac{du}{d\theta} - \frac{u}{a}\frac{1}{\sin^2\theta}\right)$$

$$+ (1-\nu)\cot\theta\left(\frac{du}{a\,d\theta} - \frac{u}{a}\cot\theta\right)$$

$$+ \left(\frac{u}{a}+\frac{dw}{a\,d\theta}\right)\left[\left(\frac{u}{a}\cot\theta - \frac{w}{a}\right) + \nu\left(\frac{du}{a\,d\theta}-\frac{w}{a}\right) - \frac{pa}{2}\right] = 0.$$

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - (1+\nu)\frac{dw}{a\,d\theta} + \cot\theta\frac{du}{a\,d\theta} - \frac{u}{a}\cot^2\theta - \nu\frac{u}{a}$$

$$- \frac{pa}{2}\left(\frac{u}{a}+\frac{dw}{a\,d\theta}\right) + \left(\frac{u}{a}+\frac{dw}{a\,d\theta}\right)\left[\left(\frac{u}{a}\cot\theta-\frac{w}{a}\right)+\nu\left(\frac{du}{a\,d\theta}-\frac{w}{a}\right)\right] = 0$$

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - \left[1+\nu+\frac{pa}{2}\right]\frac{dw}{a\,d\theta} + \cot\theta\frac{du}{a\,d\theta} - \left(\cot^2\theta+\nu+\frac{pa}{2}\right)\frac{u}{a}$$

$$+\left(\frac{u}{a}+\frac{dw}{a\,d\theta}\right)\left[\left(\frac{u}{a}\cot\theta-\frac{w}{a}\right)+\nu\left(\frac{du}{a\,d\theta}-\frac{w}{a}\right)\right]=0.$$

$$\left(\frac{du}{a\,d\theta}+\frac{u}{a}\cot\theta-\frac{2w}{a}\right)(1+\nu) + pa$$

$$+\left(\frac{d^2w}{a\,d\theta^2}+\frac{du}{a\,d\theta}\right)\left[\frac{du}{a\,d\theta}-\frac{w}{a}+\nu\left(\frac{u}{a}\cot\theta-\frac{w}{a}\right)-\frac{pa}{2}\right]$$

$$+\cot\theta\left(\frac{u}{a}+\frac{dw}{a\,d\theta}\right)\left[\frac{u}{a}\cot\theta-\frac{w}{a}+\nu\left(\frac{du}{a\,d\theta}-\frac{w}{a}\right)-\frac{pa}{2}\right]=0.$$

(9)

$$\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cos\theta - \frac{2w}{a}\right)(1+\nu) \quad -\frac{pa}{2}\left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta} + \cos\theta\,\frac{u}{a} + \cos\theta\,\frac{dw}{a\,d\theta}\right)$$

~~$$\left(1+\nu - \frac{pa}{2}\right)\frac{du}{a\,d\theta} + \left(1+\nu - \frac{pa}{2}\right)\frac{u}{a}\cos\theta - \frac{pa}{2}\,\frac{d^2w}{a\,d\theta^2} - \frac{pa}{2}\cos\theta\,\frac{dw}{a\,d\theta} - (1+\nu)\frac{2w}{a}$$~~

$$\left(1+\nu - \frac{pa}{2}\right)\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cos\theta\right) - \frac{pa}{2}\,\frac{d^2w}{a\,d\theta^2} - \frac{pa}{2}\cos\theta\,\frac{dw}{a\,d\theta} - (1+\nu)\frac{2w}{a}$$

$$+ \left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta}\right)\left[\frac{du}{a\,d\theta} - \frac{w}{a} + \nu\left(\frac{u}{a}\cos\theta - \frac{w}{a}\right)\right]$$

$$+ \cos\theta\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\left[\frac{u}{a}\cos\theta - \frac{w}{a} + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] = 0.$$

Neglect the change in curvature in y-~~axis~~ direction, we have

~~$$\frac{1}{a}\frac{d^2u}{d\theta^2} - \left[1+\nu+\frac{pa}{2}\right]\frac{dw}{a\,d\theta} + \cos\theta\,\frac{du}{a\,d\theta} - \left(\cos^2\theta - \nu\,\frac{u}{a}\right)$$~~

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - \left[1+\nu+\frac{pa}{2}\right]\frac{dw}{a\,d\theta} + \cos\theta\,\frac{du}{a\,d\theta} - \left(\cos^2\theta + \nu + \frac{pa}{2}\right)\frac{u}{a} = 0.$$

$$\left(1+\nu - \frac{pa}{2}\right)\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cos\theta\right) - \frac{pa}{2}\left(\frac{d^2w}{a\,d\theta^2} + \cos\theta\,\frac{dw}{a\,d\theta}\right) - (1+\nu)\frac{2w}{a}$$

$$+ \left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta}\right)\left[\frac{du}{a\,d\theta} - \frac{w}{a} + \nu\left(\frac{u}{a}\cos\theta - \frac{w}{a}\right)\right] = 0.$$

Putting $u = \frac{d\psi}{d\theta}$, we have from the first equation by integrating 12)

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - (1+\nu)(\psi+w) - \frac{pa}{2}(\psi+w) = 0$$

Similarly, by putting $u = \frac{d\psi}{d\theta}$ into the second equation, we have

$$(1+\nu - \frac{pa}{2})\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta}\right) - \frac{pa}{2}\left(\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta}\right)$$
$$- (1+\nu)\, 2w + \left(\frac{d^2w}{a\,d\theta^2} + \frac{d^2\psi}{a\,d\theta^2}\right)\left(\frac{d^2\psi}{d\theta^2} - \frac{w}{a}(1+\nu) + \nu\cot\theta\frac{d\psi}{d\theta}\right) = 0.$$

Let $\frac{u}{a} \sim (u)$, $\frac{w}{a} \sim (w)$, $\frac{pa}{2} = \phi$.

$$\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta} + 2\psi - (1+\nu)(\psi+w) - \phi(\psi+w) = 0$$

$$(1+\nu-\phi)\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta}\right) - \phi\left(\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta}\right) - 2(1+\nu)w$$
$$+ \left(\frac{d^2w}{d\theta^2} + \frac{d^2\psi}{d\theta^2}\right)\left(\frac{d^2\psi}{d\theta^2} + \nu\cot\theta\frac{d\psi}{d\theta} - (1+\nu)w\right) = 0.$$

or $$(1+\nu)\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta}\right) - \phi\left(\frac{d^2(\psi+w)}{d\theta^2} + \cot\theta\frac{d(\psi+w)}{d\theta}\right) - 2(1+\nu)w$$
$$+ \left[\frac{d^2(\psi+w)}{d\theta^2}\right]\left[\frac{d^2\psi}{d\theta^2} + \nu\cot\theta\frac{d\psi}{d\theta} - (1+\nu)w\right] = 0.$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - [(1+\nu)+\phi]\gamma = 0 \qquad (13)$$

$$(1+\nu)[-1+\nu+\phi]\gamma - \phi\left[\frac{d^2\gamma}{d\theta^2} + \cot\theta\frac{d\gamma}{d\theta}\right] - \cancel{2(1+\nu)\gamma}$$

$$+ \frac{d^2\gamma}{d\theta^2}\left[\frac{d^2\psi}{d\theta^2} + \nu\cot\theta\frac{d\psi}{d\theta} + (1+\nu)\psi - (1+\nu)\gamma\right] = 0.$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta} + 2\psi - [1+\nu+\phi]\gamma = 0$$

$$\frac{d^2\gamma}{d\theta^2}\left[-(1-\nu)\cot\theta\frac{d\psi}{d\theta} - (1-\nu)\psi + \phi\gamma\right] + (1+\nu)[\phi-(1-\nu)]\gamma$$

$$- \phi\left[\frac{d^2\gamma}{d\theta^2} + \cot\theta\frac{d\gamma}{d\theta}\right] = 0.$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta} + 2\psi - [1+\nu+\phi]\gamma = 0$$

$$(1+\nu)[\phi-(1-\nu)]\gamma - \frac{d^2\gamma}{d\theta^2}\left[(1-\nu)\cot\frac{d\psi}{d\theta} + (1-\nu)\psi - \phi\gamma\right] - \phi\left[\frac{d^2\gamma}{d\theta^2} + \cot\frac{d\gamma}{d\theta}\right] = 0.$$

Neglecting the curvature term, we have (14)

$$H(\psi) - (1+\nu)(\psi+w) - \phi(\psi+w) = 0.$$

$$(1+\nu-\phi)\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta}\right) - \phi\left(\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta}\right) - (1+\nu)$$

$$= 0$$

$$(1+\nu-\phi)\left[H(\psi) - 2\psi\right] - \phi\left(H(w) - 2w\right) - (1+\nu)\,2w$$

Put in $\psi = \sum_{n=0}^{\infty} A_n P_n$

$$w = \sum_{n=0}^{\infty} B_n P_n$$

$$\sum_{n=0}^{\infty}\left[-A_n\lambda_n - (1+\nu+\phi)(A_n + B_n)\right]P_n = 0$$

$$\sum_{n=0}^{\infty}\Big[(1+\nu-\phi)\left[-\lambda_n A_n - 2A_n\right] - \phi\left[-\lambda_n B_n - 2B_n\right]$$

$$-2(1+\nu)B_n\Big]P_n = 0.$$

$$\sum_{n=0}^{\infty}\left[(1+\nu+\phi+\lambda_n)A_n + (1+\nu+\phi)B_n\right]P_n = 0$$

$$\sum_{n=0}^{\infty}\left[(1+\nu-\phi)(2+\lambda_n)A_n - \left\{\phi(2+\lambda_n) - 2(1+\nu)\right\}B_n\right]P_n = 0.$$

15)

The set of homogeneous equation for A_n and B_n is

$$(1+\nu+\phi+\lambda_n)A_n + (1+\nu+\phi)B_n = 0$$

$$(1+\nu-\phi)(2+\lambda_n)A_n + \{2(1+\nu) - \phi(2+\lambda_n)\}B_n = 0.$$

The determinant must be zero, so

$$(1+\nu+\phi+\lambda_n)\{2(1+\nu) - \phi(2+\lambda_n)\}$$

$$- (1+\nu+\phi)(1+\nu-\phi)(2+\lambda_n) = 0.$$

$$2(1+\nu+\phi+\lambda_n)(1+\nu) - (1+\nu+\lambda_n)(2+\lambda_n)\phi - \phi^2(2+\lambda_n)$$

$$-(2+\lambda_n)(1+\nu+\phi)(1+\nu) + (2+\lambda_n)(1+\nu)\phi + \phi^2(2+\lambda_n) = 0$$

$$2(1+\nu)^2 + 2(1+\nu)(\phi+\lambda_n) - (2+\lambda_n)\lambda_n\phi$$

$$-2(1+\nu)^2 - \lambda_n(1+\nu+\phi)(1+\nu) - 2\phi(1+\nu) = 0.$$

$$2(1+\nu)\lambda_n - (2+\lambda_n)\lambda_n\phi - \lambda_n(1+\nu)\phi - \lambda_n(1+\nu)^2 = 0$$

$$2(1+\nu)\phi + 2(1+\nu)\lambda_n$$

$$2(1+\nu)^2 + 2(1+\nu)(\phi+\lambda_n) - \lambda_n(2+\lambda_n)\phi$$

$$-2(1+\nu)^2 - 2(1+\nu)\phi - \lambda_n(1+\nu)^2 - \lambda_n\phi(1+\nu) = 0.$$

$$\lambda_n(1+\nu)[2-(1+\nu)] - \lambda_n\phi[2+\lambda_n+1+\nu] = 0.$$

$$(1+\nu)(1-\nu) - \phi[2+\lambda_n+1+\nu] = 0.$$

$$\phi = \frac{1-\nu^2}{3+\nu+\lambda_n} \qquad \text{Min. } \phi = 0$$

$$\frac{pa}{2} = \frac{qa(1-\nu^2)}{2Et} = \frac{1-\nu^2}{3+\nu+\lambda_n}$$

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$$\frac{q}{2E}\left(\frac{a}{t}\right) = \frac{1}{3+\nu+\lambda_n}$$

$$q_{cr} = \frac{2E}{3+\nu+\lambda_n}\left(\frac{t}{a}\right)$$

$$\sigma_{cr} = \frac{2E}{3+\nu+\lambda_n}\left(\frac{t}{a}\right)\left(\frac{a}{t}\right)\frac{1}{2} = \left(\frac{E}{3+\nu+\lambda_n}\right)$$

$$\theta_1 = \theta_0 + \frac{u}{a}$$

$$r = a + w = \ldots$$

$ds_1 = 1 - \ldots$ [illegible]

The original length of the element $(ds)_0 = a\,(d\theta_0)$

The new length of the element

$$= \sqrt{r^2 (d\theta)_1^2 + (dr_1)^2} = a\sqrt{\left(1+\frac{w}{a}\right)^2\left(1+\frac{1}{a}\frac{du}{d\theta}\right)^2 + \frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2}\; d\theta$$

~~Neglecting quadratic terms of deflections~~

$$= a\,d\theta \left\{ 1 + \frac{w}{a}\left(2+\frac{w}{a}\right) + \frac{1}{a}\frac{du}{d\theta}\left(2+\frac{1}{a}\frac{du}{d\theta}\right) + \frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2 \right\}^{\frac{1}{2}}$$

The distance of the element from the axis is

$a \sin\theta_0$ before deflection

The distance is $r\sin\theta_1$ after deflection

The change in length of the ring ds is

$$2\pi a\left[\left(1+\frac{w}{a}\right)\sin\left(\theta+\frac{u}{a}\right) - \sin\theta\right]$$

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The change per unit length (circumferential) (latitude)

$$= \text{Lt} \quad \frac{\left(1+\frac{w}{a}\right)\sin\left(\theta+\frac{u}{a}\right)}{\sin\theta} - 1$$

The change per unit length (meridian)

$$\left\{1+\frac{1}{a}\frac{du}{d\theta}\left(2+\frac{1}{a}\frac{du}{d\theta}\right)+\frac{w}{a}\left(2+\frac{w}{a}\right)+\frac{1}{a}\frac{dw}{d\theta}\left(\frac{1}{a}\frac{dw}{d\theta}\right)\right\}^{\frac{1}{2}}-1$$

$$\approx \frac{1}{a}\frac{du}{d\theta}\left(1+\frac{1}{2a}\frac{du}{d\theta}\right)+\frac{w}{a}\left(1+\frac{w}{2a}\right)+\frac{1}{a}\frac{dw}{d\theta}\left(+\frac{1}{2a}\frac{dw}{d\theta}\right)$$

$$-\frac{1}{2}\frac{1}{a^2}\left(\frac{du}{d\theta}\right)^2-\frac{1}{2}\frac{1}{a^2}w^2$$

$$\approx \frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2 \quad \text{[up to second order terms]}$$

$$\frac{\left(1+\frac{w}{a}\right)\sin\left(\theta+\frac{u}{a}\right)}{\sin\theta}-1$$

$$= \left(1+\frac{w}{a}\right)\left[\cos\left(\frac{u}{a}\right)+\cot\theta\,\sin\left(\frac{u}{a}\right)\right]-1$$

$$\cong \left(1+\frac{w}{a}\right)\left[1-\frac{1}{2}\left(\frac{u}{a}\right)^2+\cot\theta\cdot\left(\frac{u}{a}\right)\right]-1$$

$$\cong -\frac{1}{2}\left(\frac{u}{a}\right)^2+\cot\theta\cdot\left(\frac{u}{a}\right)+\cot\theta\cdot\left(\frac{u}{a}\right)\left(\frac{w}{a}\right)+\frac{w}{a}$$

The stress in latitude direction

$$\frac{E}{1-\nu^2}\left[\frac{w}{a}+\frac{u}{a}\cdot\cot\theta-\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{uw}{a^2}\cot\theta+\nu\left\{\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2\right\}\right]$$

$$-\frac{pa}{2t}$$

The stress in meridian direction

$$\frac{E}{1-\nu^2}\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2+\nu\left\{\frac{w}{a}+\frac{u}{a}\cdot\cot\theta-\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{uw}{a^2}\cot\theta\right\}\right]$$

$$-\frac{pa}{2t}$$

The strain energy retaining only terms up to second order

$$2\pi a^2 t\,\frac{E}{1-\nu^2}\left\{\frac{1}{2}\int\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\nu\left\{\frac{w}{a}+\frac{u}{a}\cot\theta\right\}\right]\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right]\sin\theta\,d\theta\right.$$

$$\left.+\frac{1}{2}\int\left[\frac{w}{a}+\frac{u}{a}\cot\theta+\nu\left\{\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right\}\right]\left[\frac{w}{a}+\frac{u}{a}\cot\theta\right]\sin\theta\,d\theta\right\}$$

$$-\frac{pa}{2t}2\pi a^2 t\int\left[\frac{w}{a}+\frac{u}{a}\cot\theta-\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{uw}{a^2}\cot\theta\right]\sin\theta\,d\theta$$

$$-\frac{pa}{2t}2\pi a^2 t\int\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2\right]\sin\theta\,d\theta$$

The volume under the shell

$$= \frac{2\pi a^3}{3} \int \left[1+\frac{w}{a}\right]^3 \left[\sin\theta\left[1-\frac{1}{2}\left(\frac{u}{a}\right)^2\right] + \cos\theta\left(\frac{u}{a}\right)\right]\left[1+\frac{1}{a}\frac{du}{d\theta}\right]d\theta$$

$$= \frac{2\pi a^3}{3} \int \left\{1+3\frac{w}{a}+3\left(\frac{w}{a}\right)^2\right\}\left\{\left[1-\frac{1}{2}\left(\frac{u}{a}\right)^2\right]\sin\theta + \left(\frac{u}{a}\right)\cos\theta\right\}\left\{1+\frac{1}{a}\frac{du}{d\theta}\right\}d\theta$$

$$\sim \frac{2\pi a^3}{3} \int\left[\sin\theta\left(1+3\frac{w}{a}+3\frac{w^2}{a^2}\right) - \frac{1}{2}\left(\frac{u}{a}\right)^2\sin\theta + \left(\frac{u}{a}\right)\cos\theta + 3\cos\theta\,\frac{uw}{a^2}\right]\left[1+\frac{1}{a}\frac{du}{d\theta}\right]d\theta.$$

$$\sim \frac{2\pi a^3}{3} \int\left[\sin\theta\left(3\frac{w}{a}+3\frac{w^2}{a^2}\right) - \frac{1}{2}\left(\frac{u}{a}\right)^2\sin\theta + \left(\frac{u}{a}\right)\cos\theta + 3\cos\theta\,\frac{uw}{a^2} + \frac{1}{a}\frac{du}{d\theta}\left(\sin\theta + 3\sin\theta\,\frac{w}{a} + \left(\frac{u}{a}\right)\cos\theta\right)\right]d\theta.$$

The integral to be minimized is

$$\left\{\frac{t}{a}\frac{E}{1-\nu^2}\right\}\left[\frac{1}{2}\int\left\{\left(\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right)^2+\left(\frac{w}{a}+\frac{u}{a}\cot\theta\right)^2+2\nu\left(\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right)\left(\frac{w}{a}+\frac{u}{a}\cot\theta\right)\right\}\right.$$

$$\sin\theta\, d\theta$$

$$-\frac{p}{2}\int\left\{\frac{2w}{a}+\cot\theta\,\frac{u}{a}+\frac{1}{a}\frac{du}{d\theta}+\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2+\frac{uw}{a^2}\cot\theta\right\}\sin\theta\, d\theta$$

$$-\frac{p}{3}\int\left\{\frac{3w}{a}+3\left(\frac{w}{a}\right)^2-\frac{1}{2}\left(\frac{u}{a}\right)^2+\cot\theta\left(\frac{u}{a}\right)+3\cot\theta\,\frac{uw}{a^2}\right.$$

$$\left.+\frac{1}{a}\frac{du}{d\theta}\left(1+3\frac{w}{a}+\cot\theta\,\frac{u}{a}\right)\right\}\sin\theta\, d\theta \qquad \pi_1$$

The integral to be minimized is

$$\left(\frac{t}{a}\frac{E}{1-\nu^2}\right)\left[\frac{1}{2}\int\left\{\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right]^2+\left[\frac{w}{a}+\frac{u}{a}\cot\theta\right]^2+2\nu\left(\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right)\left(\frac{w}{a}+\frac{u}{a}\cot\theta\right)\right\}\sin\theta\, d\theta\right.$$

$$-p\int\left\{\frac{2w}{a}+\frac{5}{6}\cot\theta\,\frac{u}{a}+\frac{5}{6}\,\frac{1}{a}\frac{du}{d\theta}-\frac{5}{12}\left(\frac{u}{a}\right)^2+\frac{1}{2a^2}\left(\frac{dw}{d\theta}\right)^2\right.$$

$$\left.+\frac{3}{2}\frac{uw}{a^2}\cot\theta+\cot\theta\,\frac{1}{a^2}u\frac{du}{d\theta}\right\}\sin\theta\, d\theta.$$

$$-p\int\left\{\frac{w}{a}+\left(\frac{w}{a}\right)^2-\frac{1}{6}\left(\frac{u}{a}\right)^2+\frac{1}{3}\left(\frac{u}{a}\right)\cot\theta+\frac{uw}{a^2}\cot\theta+\frac{1}{a}\frac{du}{d\theta}\left(\frac{1}{3}+\frac{w}{a}+\cot\theta\,\frac{u}{3a}\right)\right\}$$

$$\sin\theta\, d\theta.$$

Bending without extension in meridian

We have the differential equations

$$\frac{dN_x'}{d\theta} + (N_x' - N_y')\cot\theta - Q_x - \frac{qa}{2}\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) = 0$$

$$\frac{dQ_x}{d\theta} + Q_x\cot\theta + N_x' + N_y' + qa\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta - \frac{2w}{a}\right)$$

$$- \frac{qa}{2}\left(\frac{du}{a\,d\theta} + \frac{d^2w}{a\,d\theta^2}\right) - \frac{qa}{2}\cot\theta\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) = 0$$

$$\frac{dM_x}{d\theta} + (M_x - M_y)\cot\theta - Q_x a = 0.$$

Putting $\quad w = \frac{du}{d\theta}$

$$N_x' = \frac{Eh}{1-\nu^2}\left\{\nu\left(\frac{u\cot\theta}{a} - \frac{du}{a\,d\theta}\right)\right\}$$

$$N_y' = \frac{Eh}{1-\nu^2}\left\{\frac{u\cot\theta}{a} - \frac{du}{a\,d\theta}\right\}$$

$$M_x' = -\frac{D}{a^2}\left[\frac{du}{d\theta} + \frac{d^3u}{d\theta^3} + \nu\left(u + \frac{d^2u}{d\theta^2}\right)\cot\theta\right]$$

$$M_y = -\frac{D}{a^2}\left[\left(u + \frac{d^2u}{d\theta^2}\right)\cot\theta + \nu\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\right]$$

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$$\frac{dN_x'}{d\theta} + (N_x' - N_y')\cot\theta - Q_x - \frac{qa}{2}\left(\frac{u}{a} + \frac{1}{a}\frac{d^2u}{d\theta^2}\right) = 0.$$

$$\frac{dQ_x}{d\theta} + Q_x\cot\theta + N_x' + N_y' + qa\left[-\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta - \frac{1}{2}\frac{du}{a\,d\theta} - \frac{1}{2}\frac{d^3u}{a\,d\theta^3}\right.$$

$$\left. - \frac{\cot\theta}{2}\left(\frac{u}{a} + \frac{d^2u}{a\,d\theta^2}\right)\right] = 0.$$

or

$$\frac{dQ_x}{d\theta} + Q_x\cot\theta + N_x' + N_y' + \frac{qa}{2}\left[\cot\theta\left(\frac{u}{a} - \frac{d^2u}{a\,d\theta^2}\right)\right.$$

$$\left. - 3\frac{du}{a\,d\theta} - \frac{d^3u}{a\,d\theta^3}\right] = 0.$$

$$Q_x = \frac{1}{a}\left\{\frac{dM_x}{d\theta} + (M_x - M_y)\cot\theta\right\}$$

$1 + \cot^2\theta = \csc^2\theta$

~~$\frac{1}{a}\left\{\frac{d^2M_x}{d\theta^2} + \left(\frac{dM_x}{d\theta} - \frac{dM_y}{d\theta}\right.\right.$~~

$$= -\frac{D}{a^3}\left\{\frac{d^2u}{d\theta^2} + \frac{d^4u}{d\theta^4} + \nu\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\cot\theta - \nu\left(u + \frac{d^2u}{d\theta^2}\right)\csc^2\theta\right.$$

$$\left. + (1-\nu)\cot\theta\left(u - \frac{du}{d\theta} + \frac{d^2u}{d\theta^2} - \frac{d^3u}{d\theta^3}\right)\right\}$$

$$Q_x = -\frac{D}{a^3}\left\{\frac{d^2u}{d\theta^2} + \frac{d^4u}{d\theta^4} + \nu\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\cot\theta - \nu\left(u + \frac{d^2u}{d\theta^2}\right)\csc^2\theta\right.$$

$$\left. - \cot\theta\left(u - \frac{du}{d\theta} + \frac{d^2u}{d\theta^2} - \frac{d^3u}{d\theta^3}\right) + \nu\left(u - \frac{du}{d\theta} + \frac{d^2u}{d\theta^2} - \frac{d^3u}{d\theta^3}\right)\cot\theta\right\}$$

$$= -\frac{D}{a^3}\left\{\frac{d^2u}{d\theta^2} + \frac{d^4u}{d\theta^4} - \nu\left(u + \frac{d^2u}{d\theta^2}\right)\csc^2\theta - (1-\nu)\cot\theta\left(u + \frac{d^2u}{d\theta^2}\right)\right.$$

$$\left. + \cot\theta\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\right\}$$

$$-\frac{D}{a^3}\Bigg[\frac{d^3u}{d\theta^3}+\frac{d^5u}{d\theta^5}-\nu\left(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\right)\csc^2\theta+\nu\left(u+\frac{d^2u}{d\theta^2}\right)\csc^2\theta\cdot\cot\theta \qquad 24)$$

$$+(1-\nu)\left(u+\frac{d^2u}{d\theta^2}\right)-(1-\nu)\cot\theta\left(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\right)$$

$$-\left(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\right)+2\cot\theta\left(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\right)$$

$$+\cot\theta\left(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\right)-\nu\left(u+\frac{d^2u}{d\theta^2}\right)\csc^2\theta\cot\theta$$

$$-(1-\nu)\cot^2\theta\left(u+\frac{d^2u}{d\theta^2}\right)+\cot^2\theta\left(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\right)\Bigg]$$

$$+\frac{Eh}{1-\nu^2}(1+\nu)\left[\frac{u\cot\theta}{a}-\frac{du}{a\,d\theta}\right]+\frac{qa}{2}\Big[\cot\theta\left(\frac{u}{a}-\frac{d^2u}{a\,d\theta^2}\right)$$

$$-3\frac{du}{a\,d\theta}-\frac{d^3u}{a\,d\theta^3}\Big]=0$$

$$-\frac{D}{a^3}\Bigg[\frac{d^3u}{d\theta^3}+\frac{d^5u}{d\theta^5}-(1+\nu)\left(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\right)-\left\{(1-\nu)\cot\theta+\nu\cot^2\theta\right\}\left(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\right)$$

$$+2\cot\theta\left(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\right)+\left\{(1-\nu)+\nu\left(\cot\theta+\cot^3\theta\right)\right\}\left(u+\frac{d^2u}{d\theta^2}\right)\Bigg]$$

$$+\frac{Eh}{1-\nu^2}(1+\nu)\left[\frac{u\cot\theta}{a}-\frac{du}{a\,d\theta}\right]+\frac{qa}{2}\Big[\cot\theta\left(\frac{u}{a}-\frac{d^2u}{a\,d\theta^2}\right)$$

$$-3\frac{du}{a\,d\theta}-\frac{d^3u}{a\,d\theta^3}\Big]=0.$$

$$-\alpha\Big[\frac{d^5u}{d\theta^5}+\frac{d^4u}{d\theta^4}-(1+\nu)\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)-\{(1-\nu)\cot\theta+\nu\cot^2\theta\}\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big) \tag{25}$$

$$+2\cot\theta\Big(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\Big)+\{(1-\nu)+\nu(\cot\theta+\cot^3\theta)\}\Big(u+\frac{d^2u}{d\theta^2}\Big)\Big]$$

$$+(1+\nu)\Big[u\cot\theta-\frac{du}{d\theta}\Big]+\phi\Big[\Big(u-\frac{d^2u}{d\theta^2}\Big)\cot\theta-3\frac{du}{d\theta}-\frac{d^3u}{d\theta^3}\Big]=0.$$

Put $\quad u=\dfrac{d\psi}{d\theta}$

$$-\alpha\Big[\frac{d^6\psi}{d\theta^6}+2\cot\theta\frac{d^5\psi}{d\theta^5}-\{\nu\csc^2\theta+(1-\nu)\cot\theta\}\frac{d^4\psi}{d\theta^4}$$

$$+\{(1-\nu)+(2+\nu)\cot\theta+\nu\cot^3\theta\}\frac{d^3\psi}{d\theta^3}$$

$$-\{(1+\nu)+(1-\nu)\cot\theta+\nu\cot^2\theta\}\frac{d^2\psi}{d\theta^2}$$

$$+\{(1-\nu)+\nu(\cot\theta+\cot^3\theta)\}\frac{d\psi}{d\theta}\Big]$$

$$+(1+\nu)\Big[\cot\theta\frac{d\psi}{d\theta}-\frac{d^2\psi}{d\theta^2}\Big]+\phi\Big[\Big(\frac{d\psi}{d\theta}-\frac{d^3\psi}{d\theta^3}\Big)\cot\theta-3\frac{d^2\psi}{d\theta^2}-\frac{d^4\psi}{d\theta^4}\Big]=0$$

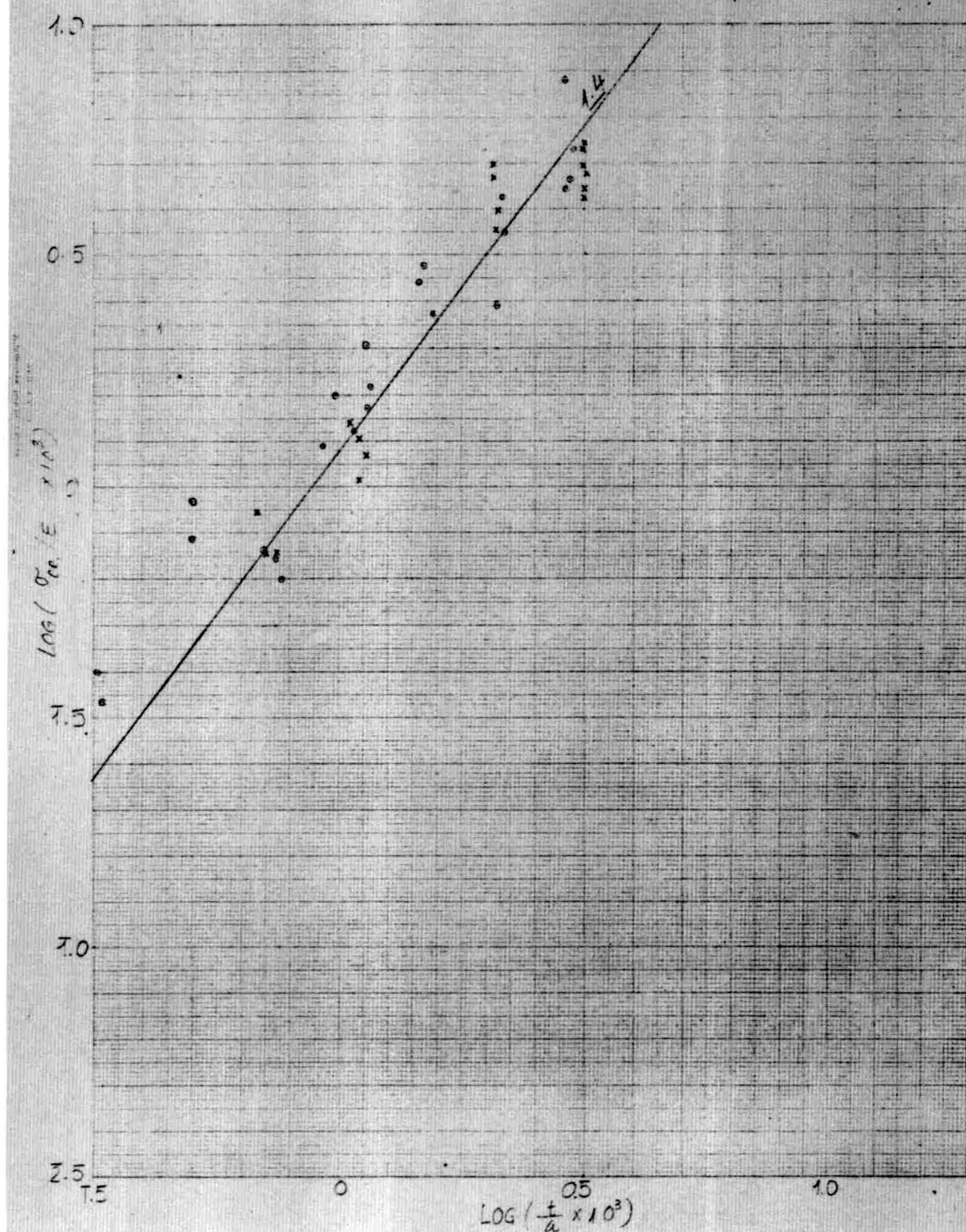
<u>Pure Bending</u> Change in curvature $\frac{1}{a}$

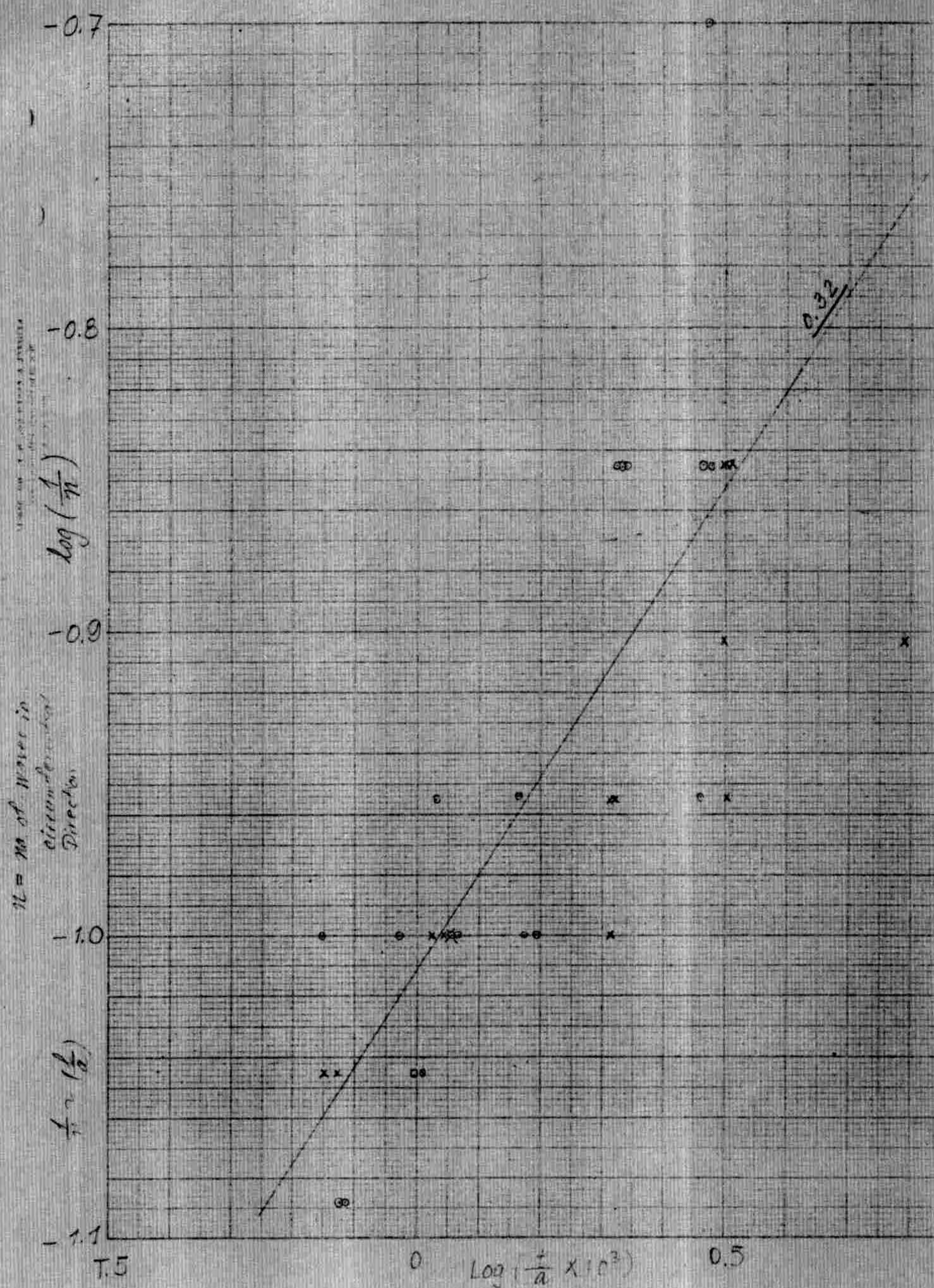
Strain energy $\sim$ (change in curvature)^2 bending stiffness $\times$ area $\sim$ $\left(\frac{1}{a}\right)^2 Et^3\cdot a^2 \sim Et^3$

Potential energy $\sim p a^3$

$p_{cr} \sim E\left(\frac{t}{a}\right)^3 \qquad \sigma_{cr} \sim E\left(\frac{t}{a}\right)^2$

	Steel			Brass			
	$\frac{t}{a}$	A	σ_{cr}/E	$\frac{t}{a}$	A	σ_{cr}/E	
1	1.015×10^{-3}	0.00512	1.484×10^{-4}	2.060×10^{-3}	0.01037	3.96×10^{-4}	27
2	0.981×10^{-3}	0.00494	1.586×10^{-4}	3.180×10^{-3}	0.00701	4.40×10^{-4}	28
3	1.556×10^{-3}	0.00334	2.355×10^{-4}	3.15×10^{-3}	0.00694	4.21×10^{-4}	29
4	1.483×10^{-3}	0.00327	2.99×10^{-4}	6.19×10^{-3}	0.00345	12.12×10^{-4}	30
5	3.012×10^{-3}	0.00168	5.36×10^{-4}	1.05×10^{-3}	0.00530	1.38×10^{-4}	31
6	2.885×10^{-3}	0.00161	7.58×10^{-4}	3.13×10^{-3}	0.00175	4.94×10^{-4}	32
7	0.765×10^{-3}	0.00386	0.63×10^{-4}	3.16×10^{-3}	0.00175	5.54×10^{-4}	33
8	0.744×10^{-3}	0.00375	0.70×10^{-4}	0.744×10^{-3}	0.00376	0.72×10^{-4}	34
9	1.162×10^{-3}	0.00257	1.65×10^{-4}	0.710×10^{-3}	0.00357	0.72×10^{-4}	35
10	1.137×10^{-3}	0.00250	2.03×10^{-4}	1.136×10^{-3}	0.00251	1.17×10^{-3}	36
11	2.175×10^{-3}	0.00121	3.54×10^{-4}	1.10×10^{-3}	0.00244	1.03×10^{-3}	37
12	2.155×10^{-3}	0.00120	4.23×10^{-4}	2.09×10^{-3}	0.01060	3.60×10^{-3}	38
13	0.932×10^{-3}	0.00470	1.23×10^{-4}	2.06×10^{-3}	0.01044	5.02×10^{-3}	39
14	1.460×10^{-3}	0.00322	2.75×10^{-4}	2.06×10^{-3}	0.01044	4.79×10^{-3}	40
15	2.906×10^{-3}	0.00162	4.46×10^{-4}	3.19×10^{-3}	0.00704	4.77×10^{-3}	41
16	0.702×10^{-3}	0.00354	0.73×10^{-4}	6.23×10^{-3}	0.00347	14.18×10^{-3}	42
17	1.071×10^{-3}	0.00237	1.32×10^{-4}	0.68×10^{-3}	0.00343	0.88×10^{-3}	43
18	2.970×10^{-3}	0.00166	4.60×10^{-4}	1.10×10^{-3}	0.00244	1.28×10^{-3}	44
19	2.110×10^{-3}	0.00118	2.46×10^{-4}	2.09×10^{-3}	0.01052	3.56×10^{-3}	45
20	0.5×10^{-3}		0.15×10^{-4}	3.15×10^{-3}	0.00696	5.40×10^{-3}	46
21	"		0.77×10^{-4}	6.34×10^{-3}	0.00353	14.74×10^{-3}	47
22	0.33×10^{-3}		0.342×10^{-4}				48
23	0.322×10^{-3}		0.400×10^{-4}				49
24							50
25							51
26							52


28)
1.0
0.5
1.5
1.0
2.5
1.4
1.5
0
0.5
1.0


-0.7
-0.8
-0.9
-1.0
-1.1
0.32
log (1/n)
n = no. of waves in circumferential Direction
T.5
0
0.5

1)

Important Papers on Curved Sheets

K. Marguerre: Über die Behandlung von Stabilitätsproblemen mit Hilfe der energetischen Methode
ZAMM 18: 57–73 (1938)

✓ K. Marguerre & E. Trefftz: Über die Tragfähigkeit eines Plattenstreifens nach Überschreiten der Beullast
ZAMM 17: 85–100 (1937)

K. v. Sanden; F. Tölke: Über Stabilitätsprobleme dünner kreiszylindrischer Schalen. Ing.-Arch. 3: 24-66 (1932)

W. Flügge: Die Stabilität der Kreiszylinderschale
Ing.-Arch. 3: 463-506 (1932)

✓ K. Marguerre: Die mittragende Breite der gedrückten Platte
Luftfahrtforschung (1937), 14: 124.

T. E. Schunck: Zur Knickfestigkeit schwach gekrümmter zylindrischer Schalen Ing.-Arch. 4: 394-414 (1933)

✓ N. J. Hoff: Instability of Monocoque Structures in Pure Bending
Journal of Roy. Ae. Soc. (1938) 42: 291 – 345

2)

✓ L. H. Donnell: A New Theory for the Buckling of Thin Cylinders Under Axial Compression & Bending
Transc of ASM.E 795–806 (1934)

R. V. Southwell: On the General Theory of Elastic Stability
Transc. of Roy. Soc. of London 213: 187–244 (1914)

✓ J. L. Taylor: Stability of a Monocoque in Compression
R&M 1679 (1936)

✓ H. Ebener; H. Köller: Zur Berechnung des Kraftverlaufes in versteiften Zylinderschalen
Lff. 14: 607–626 (1937) TM 866

✓ E. Schapitz; G. Krümling: Belastungsversuche mit einer versteiften Kreiszylinderschale bei Krafteinleitung an einzelnen Punkten
Lff. 14: 593–606 (1937) TM 864

✓ H. Wagner: Einiges über schalenförmiges Flugzeug-Bauteiles Lff. 13: 281–292 (1936) TM 817

✓ O. S. Heck; H. Ebener: Formeln und Berechnungsverfahren für die Festigkeit von Platten- und ~~Schalenkon~~ Schalenkonstruktionen im Flugzeugbau
Lff 12: 211–222 (1935) TM 785

A. J. Sutton Pippard: Distortion of Thin Tubes under Flexure 3.
RM 1465 (1932)

✓ W. Kaufmann: Bemerkungen zur Stabilität dünnwandiger, kreiszylindrischer Schalen oberhalb der Proportionalitätsgrenze Ing.-Arch. 6: 419-430 (1935)

✓ J. W. Geckeler: Plastisches Knicken der Wandung von Hohlzylindern und einige andere Faltungserscheinungen an Schalen und Blechen
ZAMM 8: 341-352 (1928)

✓ W. Kaufmann: Plastisches Knicken dünnwandiger Hohlzylinder infolge axialer Belastung
Ing.-Arch. 6: 334-337 (1935)

✓ W. Kaufmann: Über unelastisches Knicken rechteckiger Platten
Ing.-Arch. 7: 156-165 (1936)

✓ A. Kromm; K. Marguerre: Verhalten eines von Schub- und Druckkräften beanspruchten Plattenstreifens oberhalb der Beulgrenze LFF. 14: 627-639 (1937)
TM 870

R. Barbré: Stabilität gleichmäßig gedrückter Rechteckplatten mit Längs oder Quersteifen Ing.-Arch. 8: 117–150 (1937) [GALCIT Trans.] 4)

A. Pflüger: Stabilität dünner Kegelschalen Ing.-Arch. 8: 151–172 (1937)

W. A. Wenzek: Die mittragende Breite nach dem Ausknicken bei krummen Blechen Lff. 15: 340–344 (1938)

X L. Föppl: Eine neue elastische Materialkonstante Ing.-Arch. 7: 229–236 (1936)

Section 2

Donnell's Equation

The Buckling of Circular Cylindrical Shell

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<u>Donnel's Equation</u> (Cong. of Applied Mech.)

$$\frac{Et^2}{12(1-\mu^2)}\left\{r^2\nabla^8 w+\frac{2}{r^6}\frac{\partial^6 w}{\partial\theta^6}+\frac{1}{r^6}\frac{\partial^4 w}{\partial\theta^4}\right\}+E\frac{\partial^4 w}{\partial x^4}=-\sigma\frac{\partial^2}{\partial x^2}\left[r^2\nabla^4 w-\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right]$$

Put $w = \sin n\theta \sin \frac{2\pi x}{l}$

Wave length in axial direction $= l$, in circumferential direction $= \frac{2\pi}{n} r = m$, $\quad \frac{2\pi}{m} = \frac{n}{r}$

$$\frac{t^2}{12(1-\mu^2)}\left[r^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{n}{r}\right)^2\right\}^4-2\left(\frac{n}{r}\right)^6+\frac{1}{r^2}\left(\frac{n}{r}\right)^4\right]+\left(\frac{2\pi}{l}\right)^4=\frac{\sigma}{E}\left[r^2\left(\frac{2\pi}{l}\right)^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{n}{r}\right)^2\right\}^2+\frac{r}{l}\right.$$

or
$$\frac{t^2}{12(1-\mu^2)}\left[r^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{2\pi}{m}\right)^2\right\}^4-2\left(\frac{2\pi}{m}\right)^6+\frac{1}{r^2}\left(\frac{2\pi}{m}\right)^4\right]+\left(\frac{2\pi}{l}\right)^4$$

$$=\frac{\sigma}{E}\left[r^2\left(\frac{2\pi}{l}\right)^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{2\pi}{m}\right)^2\right\}^2+\left(\frac{2\pi}{l}\right)^2\left(\frac{2\pi}{m}\right)^2\right]$$

Put $\quad r^2 = R^2\,\dfrac{t^2}{12(1-\mu^2)} \qquad \left(\dfrac{2\pi}{l}\right)^2 = \dfrac{\alpha}{\dfrac{t^2}{12(1-\mu^2)}}$

$$\left(\frac{2\pi}{m}\right)^2 = \frac{\beta}{\dfrac{t^2}{12(1-\mu^2)}}$$

$$R^2(\alpha+\beta)^4-2\beta^3+\frac{\beta^2}{R^2}+\alpha^2=\frac{\sigma}{E}\left[R^2\alpha(\alpha+\beta)^2+\alpha\beta\right]$$

or
$$\frac{\sigma}{E}=\frac{R^2(\alpha+\beta)^4-2\beta^3+\frac{\beta^2}{R^2}+\alpha^2}{R^2\alpha(\alpha+\beta)^2+\alpha\beta}$$

Let $\left(\frac{m}{l}\right)^2 = \lambda = \frac{\alpha}{\beta}$ or $\alpha = \lambda\beta$. 2)

$$\frac{\sigma}{E} = \frac{R^2\beta^2(1+\lambda)^4 - 2\beta + \frac{1}{R^2} + \lambda^2}{R^2\lambda\beta(1+\lambda)^2 + \lambda}$$

$$\left\{R^2\lambda\beta(1+\lambda)^2 + \lambda\right\}\left\{2R^2(1+\lambda)^4\beta - 2\right\}$$

$$- \left\{R^2\beta^2(1+\lambda)^4 - 2\beta + \frac{1}{R^2} + \lambda^2\right\} R^2\lambda(1+\lambda)^2 = 0$$

$$R^4\lambda(1+\lambda)^6\beta^2 + 2R^2\lambda(1+\lambda)^4\beta - \left\{\lambda(1+\lambda)^2 + 2\lambda + R^2\lambda^3(1+\lambda)^2\right\} = 0.$$

$$\lambda \sim 1, \qquad R \gg 1,$$

$$R^2(1+\lambda)^4\beta^2 + 2(1+\lambda)^2\beta - \lambda^2 = 0.$$

$$\beta^2 + \frac{2}{R^2(1+\lambda)^2}\beta - \frac{\lambda^2}{R^2(1+\lambda)^4} = 0$$

$$\beta = -\frac{1}{R^2(1+\lambda)^2} + \sqrt{\frac{1}{R^4(1+\lambda)^4} + \frac{\lambda^2}{R^2(1+\lambda)^4}}$$

$$= \frac{1}{R^2(1+\lambda)^2}\left[\sqrt{1+\lambda^2R^2} - 1\right]$$

$$\cong \frac{\lambda}{R(1+\lambda)^2}$$

$$\frac{\sigma}{E} = \frac{2\lambda^2 + \frac{1}{R^2} - 2[(1+\lambda)^2+1]\beta}{R^2\lambda(1+\lambda)^2\beta + \lambda}$$

$$= \frac{2\lambda^2 + \frac{1}{R^2} - \frac{2\lambda[(1+\lambda)^2+1]}{R(1+\lambda)^2}}{R\lambda^2 + \lambda} \cong \frac{2}{R}$$

$$\frac{\sigma}{E} = \frac{2}{\left(\frac{r}{t}\right)\sqrt{12(1-\mu^2)}}$$

$$\sigma = \frac{E}{\sqrt{3(1-\mu^2)}}\left(\frac{t}{r}\right)$$

Nothing New.

$$R^4(1+\lambda)^6\beta^2+2R^2(1+\lambda)^4\beta-\{(3+2\lambda+\lambda^2)+R^2\lambda^2(1+\lambda)^2\}=0 \qquad 4)$$

$$\beta^2+\frac{2}{R^2(1+\lambda)^2}\beta-\left\{\frac{(3+2\lambda+\lambda^2)}{R^4(1+\lambda)^6}+\frac{\lambda^2}{R^2(1+\lambda)^4}\right\}=0$$

$$\beta=-\frac{1}{R^2(1+\lambda)^2}+\sqrt{\frac{1}{R^4(1+\lambda)^4}+\frac{\lambda^2}{R^2(1+\lambda)^4}+\frac{3+2\lambda+\lambda^2}{R^4(1+\lambda)^6}}$$

$$=\frac{1}{R^2(1+\lambda)^2}\left[\sqrt{1+\lambda^2R^2+\frac{3+2\lambda+\lambda^2}{(1+\lambda)^2}}-1\right]$$

$$\frac{\sigma}{E}=\frac{R^2(1+\lambda)^4\beta^2-2\beta+\frac{1}{R^2}+\lambda^2}{R^2\lambda(1+\lambda)^2\beta+\lambda}$$

$$=\frac{\frac{(3+2\lambda+\lambda^2)}{R^2(1+\lambda)^2}+2\lambda^2-2[(1+\lambda)^2+1]\beta+\frac{1}{R^2}}{R^2\lambda(1+\lambda)^2\beta+\lambda}$$

$$\boxed{\frac{\sigma}{E}=\frac{\frac{(4+4\lambda+2\lambda^2)}{R^2(1+\lambda)^2}+2\lambda^2-2(2+2\lambda+\lambda^2)\beta}{\lambda[R^2(1+\lambda)^2\beta+1]}}$$

$$\frac{\sigma}{E} = \frac{\frac{(4+4\lambda+2\lambda^2)}{R^2(1+\lambda)^2} + 2\lambda^2 - 2(2+2\lambda+\lambda^2)\beta}{\lambda\sqrt{1+\lambda^2R^2+\frac{3+2\lambda+\lambda^2}{(1+\lambda)^2}}}$$

tw $\lambda = 1$

$$\frac{\sigma}{E} = \frac{\frac{9}{4}\frac{1}{R^2} + 2 - 10\left\{\frac{\sqrt{1+R^2+\frac{5}{4}} - 1}{4R^2}\right\}}{\sqrt{1+R^2+\frac{5}{4}}}$$

$$\frac{\sigma}{E} = \frac{\frac{1}{4R^2}\left[19 - 10\sqrt{1+R^2+\frac{5}{4}}\right] + 2}{\sqrt{1+R^2+\frac{5}{4}}} = \frac{\frac{19}{4R^2}+2}{\sqrt{1+R^2+\frac{5}{4}}} - \frac{5}{2R^2}$$

$$\boxed{\frac{\sigma}{E} = \frac{2+\frac{19}{4R^2}}{\sqrt{R^2+\frac{9}{4}}} - \frac{5}{2R^2}}$$

$\lambda = 100$

$$\frac{\sigma}{E} = \frac{\frac{10403}{R^2\,10201} + 20000 - 2\times 10202\,\frac{\sqrt{1+10000R^2+\frac{10202}{10201}} - 1}{R^2\,10201}}{100\sqrt{1+10000R^2+\frac{10202}{10201}}}$$

$$\simeq \frac{\frac{1.020}{R^2} + 20000 - \frac{200}{R}}{10000R} = \frac{2}{R} - \frac{0.02}{R^2} + \frac{0.000102}{R^3}$$

$$= \frac{2}{R}\left(1 - \frac{0.01}{R} + \frac{0.000051}{R^2}\right)$$

$\lambda = 0.1$

$$\frac{\sigma}{E} = \frac{\frac{4.42}{1.21}\frac{1}{R^2} + 0.02 - 4.42\,\frac{\sqrt{1+0.01R^2+\frac{3.21}{1.21}} - 1}{1.21\,R^2}}{0.1\sqrt{1+0.01R^2+\frac{2.21}{1.21}}}$$

$$= \frac{\frac{3.6540}{R^2} + 0.02 - \frac{3.6529}{R^2}\left\{\sqrt{3.652+0.01R^2} - 1\right\}}{0.1\sqrt{2.8264+0.01R^2}}$$

If $R = 100$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.036540 + 2 - 0.036529 \times 10.80}{0.1 \times 11.80}\right]$$ 0.422

$R = 300$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.01218 + 6 - 0.012176 \times 29.048}{0.1 \times 30.048}\right]$$ $K = 0.570$

$R = 1000$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.0028264 + 20 - 0.0036529 \times 100}{0.1 \times 101}\right] =$$ .595

$R = 10.$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.365 + 0.2 - 0.36529 \times 1.164}{0.1 \times 2.964}\right]$$ .214

$\lambda = 0.01$ (6)

$$\frac{\sigma}{E} = \frac{\frac{7.0402}{1.0201}\frac{1}{R^2} + 0.0002 - 4.0402\left\{\frac{\sqrt{1+\frac{3.0201}{1.0201}+0.0001R^2}-1}{1.0201\,R^2}\right\}}{0.01\sqrt{1+\frac{2.0201}{1.0201}+0.0001R^2}}$$

for. $R = 100$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.0365\text{4}0 + 0.0\;\;2 - \overset{3.960}{\frac{4.0402}{1.0201}}\,0.0122\text{6}}{0.01 \times 2.2361}\right]$$

~~$= \frac{1}{R}\left[\frac{0.010391}{0.01 \times 1.9951}\right] = \frac{2}{R} \times 0.2604.$~~

$k = 0.1578$ (crossed out; 1.8 written above)

$R = 300$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.01218 + 0.06 - \frac{4.0402}{1.0201} \times \frac{2.6603}{300}}{0.01 \times 3.6060}\right]$$

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$$= \frac{1}{R}\;\frac{0.11676}{0.0163\text{7}613}$$

0.3275

$R = 1000$

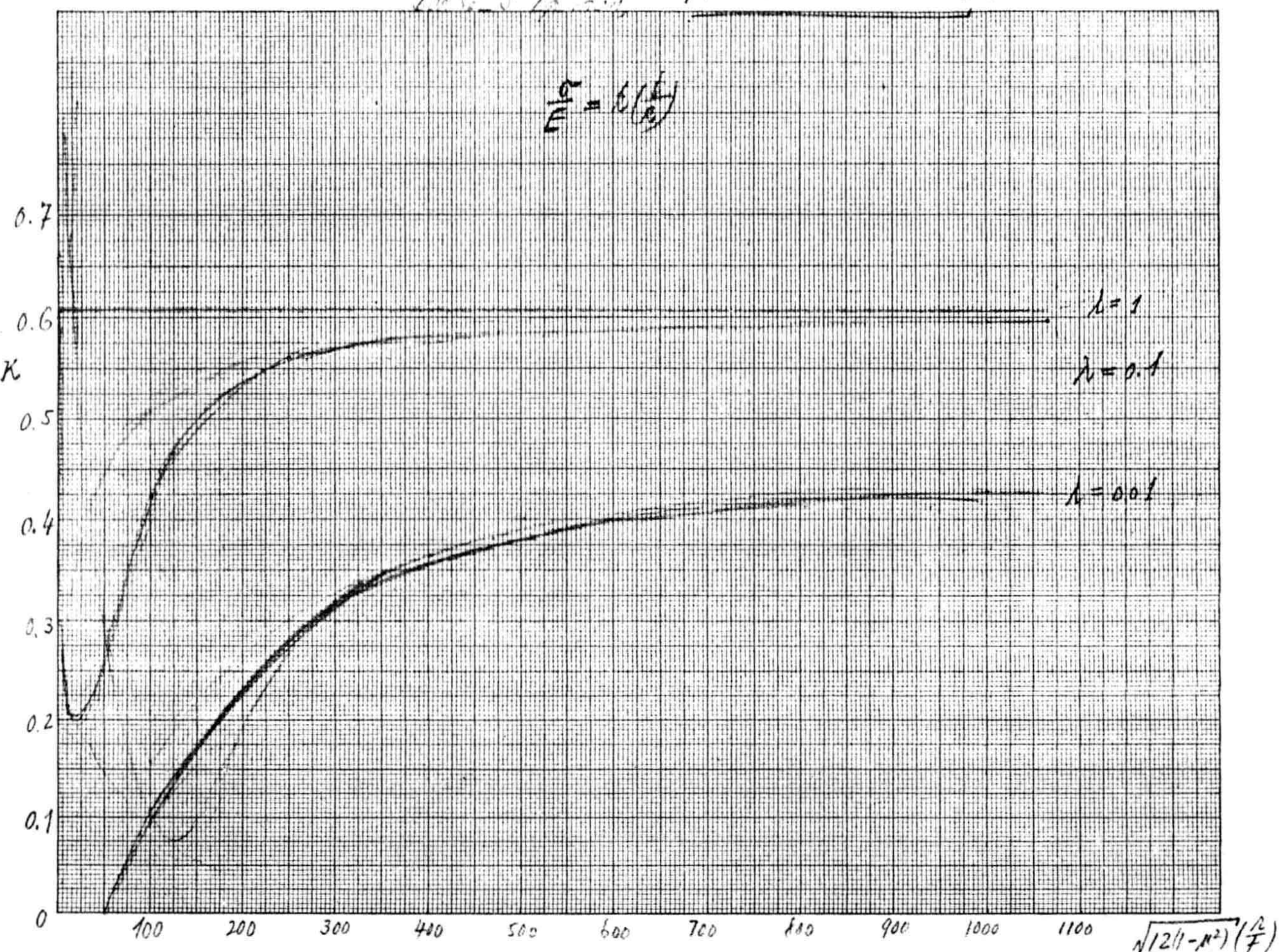
$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.0029803 + 0.2 - 3.960 \times 0.01049}{0.01 \times 11.49}\right] =$$

0.426

$R = 50$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.07308 + 0.01 - 3.960 \times \frac{1.052}{50}}{0.01 \times 2.052}\right]$$

~ 0



√λ = ratio of WAVE LENGTHS
λ = 1
λ = 0.1
λ = 0.01
K
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
100
200
300
400
500
600
700
800
900
1000
1100
$\sqrt{12(1-\mu^2)}\left(\frac{R}{t}\right)$

Section 3

Preliminary Calculation of Circular Cylinder (I)

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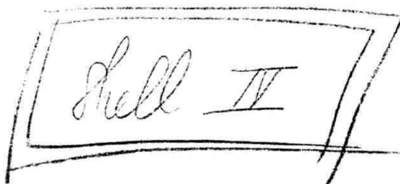

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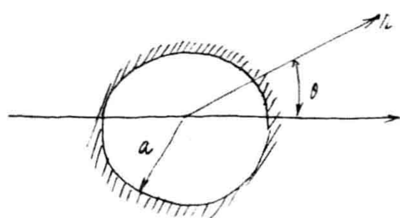
Preliminary Calculation of Circular Cylinder

PART (I)

The Increase in Strain Energy Due to the Presence of a Circular Hole

We have from Southwell's "Elasticity", p. 386, the following relations:



$$\widehat{rr} = \frac{1}{2}T\left(1-\frac{a^2}{r^2}\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]$$

$$\widehat{\theta\theta} = \frac{1}{2}T\left[1+\frac{a^2}{r^2}-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$\widehat{r\theta} = -\frac{1}{2}T\left(1-\frac{a^2}{r^2}\right)\left(1+3\frac{a^2}{r^2}\right)\sin 2\theta$$

$$\widehat{rr}_0 = \frac{1}{2}T\left(1+\cos 2\theta\right)$$

$$\widehat{\theta\theta}_0 = \frac{1}{2}T\left(1-\cos 2\theta\right)$$

$$\widehat{r\theta}_0 = -\frac{1}{2}T\sin 2\theta$$

~~If we assume that $\mu=0$, i.e., vanishing Poisson's ratio,~~ then

In case of plane stress,

strain energy per unit area $= \frac{t}{E}\left\{\frac{\widehat{rr}^2+\widehat{\theta\theta}^2}{2} - \nu\,\widehat{rr}\,\widehat{\theta\theta} + (1+\nu)\,\widehat{\theta r}^2\right\}$

Thus $\left[\dfrac{\widehat{rr}^2+\widehat{\theta\theta}^2}{2} - \nu\,\widehat{\theta\theta}\,\widehat{rr} + (1+\nu)\,\widehat{\theta r}^2\right] - \left[\dfrac{\widehat{rr}_0^{\,2}+\widehat{\theta\theta}_0^{\,2}}{2} - \nu\,\widehat{\theta\theta}_0\,\widehat{rr}_0 + (1+\nu)\,\widehat{\theta r}_0^{\,2}\right]$ $\underline{\underline{263}}$

$$= \frac{1}{4}T^2\Bigg\{\frac{1}{2}\Big[\left(1-\frac{a^2}{r^2}\right)^2\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]^2 + \left[1+\frac{a^2}{r^2}-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]^2$$

$$- (1+\cos 2\theta)^2 - (1-\cos 2\theta)^2\Big]$$

$$-\nu\Big[\left(1-\frac{a^2}{r^2}\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]\left[1+\frac{a^2}{r^2}-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right] - (1+\cos 2\theta)(1-\cos 2\theta)\Big]$$

$$+ (1+\nu)\Big[\left(1-\frac{a^2}{r^2}\right)^2\left(1+3\frac{a^2}{r^2}\right)^2\sin^2 2\theta - \sin^2 2\theta\Big]\Bigg\}$$

$$= \left(\frac{T}{2}\right)^2\Bigg\{\frac{1}{2}\Big[\frac{a^2}{r^2}\left(\frac{a^2}{r^2}-2\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]^2 + 3\frac{a^2}{r^2}\cos 2\theta\left(3\frac{a^2}{r^2}\cos 2\theta - 2(1+\cos 2\theta)\right)$$

$$+ \frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\cos 2\theta\right)\left[\frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)\cos 2\theta + 2(1-\cos 2\theta)\right]\Big]$$

$$-\nu\Big[-\frac{a^2}{r^2}\left\{(1+\cos 2\theta) - 3\frac{a^2}{r^2}\cos 2\theta\right\}\left\{(1-\cos 2\theta)+\frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)\cos 2\theta\right\}$$

$$+ \frac{a^2}{r^2}\cos 2\theta\left\{(1+\cos 2\theta)\left(1-\frac{3a^2}{r^2}\right) - 3(1-\cos 2\theta)\right\} - 3\left(1-\frac{3a^2}{r^2}\right)\frac{a^4}{r^4}\cos^2 2\theta\Big]$$

$$+ (1+\nu)\Big[\sin^2 2\theta\left\{\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right)+1\right\}\frac{a^2}{r^2}\left(2-3\frac{a^2}{r^2}\right)\Big]\Bigg\} = F(r,\theta)\left(\frac{T}{2}\right)^2$$

$$F(r,\theta) = \frac{1}{2}\left\{ \frac{a^2}{r^2}\left(\frac{a^2}{r^2}-2\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]^2 + 3\frac{a^2}{r^2}\left[3\frac{a^2}{r^2}\cos^2 2\theta - 2(\cos 2\theta + \cos^2 2\theta)\right]\right.$$

$$\left. + \frac{a^2}{r^2}\left[\frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)\cos 2\theta + 2(1-\cos 2\theta)\right] - \frac{3a^4}{r^4}\left[\frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)\cos^2 2\theta + 2(\cos 2\theta - \cos^2 2\theta)\right]\right\}$$

$$-\nu\left\{-\frac{a^2}{r^2}\left[\sin^2 2\theta - 3\frac{a^2}{r^2}(\cos 2\theta - \cos^2 2\theta) + \frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)(\cos 2\theta + \cos^2 2\theta)\right]\right.$$

$$\left. + \frac{a^2}{r^2}\left[\left(1-\frac{3a^2}{r^2}\right)(\cos 2\theta + \cos^2 2\theta) - 3(\cos 2\theta - \cos^2 2\theta)\right] - 3\left(1-\frac{3a^2}{r^2}\right)\frac{a^4}{r^4}\cos^2 2\theta\right\}$$

$$+ (1+\nu)\left\{\sin^2 2\theta \; \frac{a^2}{r^2}\left(2-3\frac{a^2}{r^2}\right)\left[\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right)+1\right]\right\}$$

Thus

$$\int_0^{2\pi} F(r,\theta)\,d\theta = \frac{\pi}{2}\left\{\frac{a^2}{r^2}\left(\frac{a^2}{r^2}-2\right)\left[2+\left(1-3\frac{a^2}{r^2}\right)^2\right] + 3\frac{a^2}{r^2}\left[3\frac{a^2}{r^2}-2\right]\right.$$

$$\left. + \frac{a^2}{r^2}\cdot 4 - \frac{3a^4}{r^4}\left[\frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)-2\right]\right\}$$

$$-\nu\pi\left\{-\frac{a^2}{r^2}\left[1+3\frac{a^2}{r^2}+\frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)\right] + \frac{a^2}{r^2}\left[\left(1-\frac{3a^2}{r^2}\right)+3\right] - 3\left(1-\frac{3a^2}{r^2}\right)\frac{a^4}{r^4}\right\}$$

$$+ \pi(1+\nu)\,\frac{a^2}{r^2}\left(2-3\frac{a^2}{r^2}\right)\left[2+2\frac{a^2}{r^2}-\frac{3a^4}{r^4}\right]\Bigg\}$$

Putting $\frac{a}{r} = \frac{1}{R}$

$$= \frac{\pi}{2}\left\{\frac{1}{R^2}\left(\frac{1}{R^2}-2\right)\left(3-6\frac{1}{R^2}+9\frac{1}{R^4}\right) + 3\frac{1}{R^2}\left(3\frac{1}{R^2}-2\right) + 4\frac{1}{R^2} - 3\frac{1}{R^4}\left[\frac{1}{R^2}-\frac{3}{R^4}-2\right]\right\}$$

$$-\nu\pi\left\{-\frac{1}{R^2}\left[1+4\frac{1}{R^2}-\frac{3}{R^4}\right] + \frac{1}{R^2}\left[4-\frac{3}{R^2}\right] - 3\frac{1}{R^4}\left(1-\frac{3}{R^2}\right)\right\}$$

$$+\pi(1+\nu)\,\frac{1}{R^2}\left(2-3\frac{1}{R^2}\right)\left(2+\frac{2}{R^2}-\frac{3}{R^4}\right)\Bigg\}$$

If for simplicity $\nu = 0$, then

$$\frac{R^4}{\pi}\int_0^{2\pi} F(r,\theta)\,d\theta = \Big[\tfrac{1}{2}\big(3 - \tfrac{6}{R^2} + 12 + \tfrac{9}{R^4} - 18 + 9 - \tfrac{3}{R^2} + \tfrac{9}{R^4} + 6\big)$$

$$+ \big(-6 + 4 - \tfrac{6}{R^2} - \tfrac{6}{R^2} + \tfrac{9}{R^4}\big)\Big]$$

$$= \Big[4 - \frac{33}{2}\frac{1}{R^2} + 18\frac{1}{R^4}\Big]$$

Energy increase due to the presence of hole of radius "a"

$$= \int_a^\infty \int_0^{2\pi} F(r,\theta)\left(\frac{\sigma}{2}\right)^2 \frac{t}{E}\, r\, dr\, d\theta$$

$$= a^2\frac{t}{E}\left(\frac{\sigma}{2}\right)^2 \int_1^\infty \int_0^{2\pi} F(r,\theta)\,d\theta \cdot R\,dR$$

$$= \frac{a^2\pi t}{E}\left(\frac{\sigma}{2}\right)^2 \int_1^\infty \Big[\frac{4}{R^3} - \frac{33}{2}\frac{1}{R^5} + \frac{18}{R^7}\Big]\,dR$$

$$= \frac{a^2\pi t}{E}\left(\frac{\sigma}{2}\right)^2 \Big[2 - \frac{33}{8} + 3\Big] = \boxed{\frac{a^2\pi t}{E}\left(\frac{\sigma}{2}\right)^2 \frac{7}{8} = \frac{a^2\pi t\sigma^2}{E}\,\frac{7}{32}}$$

$$\frac{\partial u}{\partial r} = \frac{1}{E}\,\widehat{rr}$$

$$\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}\,\widehat{\theta\theta}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{2}{E}\,\widehat{r\theta}$$

$$\frac{\partial u}{\partial r} = \frac{\sigma}{2E}\left(1 - \frac{a^2}{r^2}\right)\left[1 + \left(1 - \frac{3a^2}{r^2}\right)\cos 2\theta\right]$$

$$= \frac{\sigma}{2E}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$u = \frac{\sigma}{2E}\left[\left(r + \frac{a^2}{r}\right) + \left(r + \frac{4a^2}{r} - \frac{a^4}{r^3}\right)\cos 2\theta + F(\theta)\right]$$

$$\frac{\partial v}{\partial \theta} = \frac{1}{E}\, r \cdot \widehat{\theta\theta} - u$$

$$= \frac{\sigma}{2E}\left\{ r + \frac{a^2}{r} - \left(r + \frac{3a^4}{r^3}\right)\cos 2\theta - \left(r + \frac{a^2}{r}\right) - \left(r + \frac{4a^2}{r} - \frac{a^4}{r^3}\right)\cos 2\theta - F(\theta)\right\}$$

$$= \frac{\sigma}{2E}\left\{ -2\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\cos 2\theta - F(\theta)\right\}$$

$$v = \frac{\sigma}{2E}\left\{ -\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta - \int F(\theta) + G(r)\right\}$$

Check.

$$\frac{\sigma}{2E}\Bigg[-2\left(1+\frac{4a^2}{r^2}-\frac{a^4}{r^4}\right)\sin 2\theta+\frac{F'(\theta)}{r}-\left(1-\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta+G'(r)$$

$$+\left(1+\frac{2a^2}{r^2}+\frac{a^4}{r^4}\right)\sin 2\theta+\frac{\int F(\theta)}{r}+\frac{G(r)}{r}\Bigg]$$

$$=\frac{\sigma}{E}\left[\left(\frac{2a^2}{r^2}+\frac{2a^4}{r^4}-1-\frac{4a^2}{r^2}+\frac{a^4}{r^4}\right)\sin 2\theta+\frac{F'(\theta)+G(r)+\int F(\theta)}{r}+G'(r)\right]$$

$$=-\frac{\sigma}{E}\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta+\frac{\sigma}{E}\Big[\qquad\qquad\Big]$$

$$\therefore \qquad \frac{v}{r}=-\frac{\sigma}{2E}\left(1+\frac{a^2}{r^2}\right)^2\sin 2\theta$$

$$\frac{u}{r}=\frac{\sigma}{2E}\left[\left(1+\frac{a^2}{r^2}\right)+\left(1+\frac{4a^2}{r^2}-\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

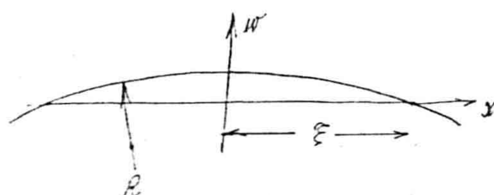
At $r=a$,

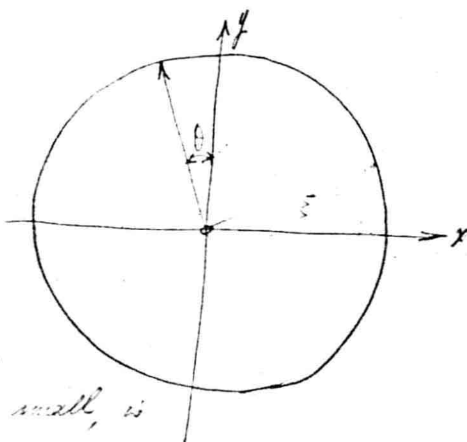
$$\boxed{\begin{aligned}\frac{v}{a}&=-\frac{\sigma}{2E}4\sin 2\theta=-\frac{2\sigma}{E}\sin 2\theta\\ \frac{u}{a}&=\frac{\sigma}{2E}(2+4\cos 2\theta)=\frac{\sigma}{E}(1+2\cos 2\theta)\end{aligned}}$$

PART (II)

Calculation of the Strain Energy of a Cylindrical Shell of Circular Plane Form with Given Displacement

1) The vertical displacement:





the original form of the shell, when the angular extension of the shell is small, is

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{\xi}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2} = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2}$$

After Buckling

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{\xi}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2} - f\left[\frac{\left(\frac{\xi\sqrt{\xi^2-y^2}}{\xi R}\right)^2 - \left(\frac{x}{R}\right)^2}{2}\right]$$

$$\left(\frac{w}{R}\right) = \frac{1}{2}\left\{\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2} - f\left[\left(\frac{a}{R}\right)^2 - \left\{\left(\frac{y}{R}\right)^2 + \left(\frac{x}{R}\right)^2\right\}\right]\right\}$$

Changing to polar coördinates

$$\left.\begin{aligned}\left(\frac{w}{R}\right)_0 &= \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}\\ \left(\frac{w}{R}\right) &= \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left[\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r^2}{R^2}\right)}{2}\right]\end{aligned}\right\} \text{Form (I)}$$

To avoid any bending moment at the edges, we approximate the circular arc with a sine wave. The amplitude of the sine wave is given by

$$R(1-\cos\beta) = 2R\sin^2\frac{\beta}{2} = R\frac{\beta^2}{2} \simeq \frac{R\left(\frac{a}{R}\right)^2}{2} \simeq \frac{a^2}{2R}$$

~~$\left(\frac{w}{R}\right)_0 = \frac{a^2}{2R}\cos\frac{\pi x}{2a}$~~

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\sin^2\theta}{2} - f\cos\frac{\pi r}{2a}$$

---

2) The equation of equilibrium of stresses in the middle plane:

$$\left.\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned}\right\}$$

These equations are satisfied by introducing the stress function φ

$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\varphi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$\sigma_r = E\left\{ \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2 \right\}$$

$$\sigma_t = E\left\{ \frac{u}{r} + \frac{\partial v}{r\,\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{1}{2r^2}\left(\frac{\partial w_0}{\partial \theta}\right)^2 \right\}$$

$$\tau_{r\theta} = \frac{E}{2}\left\{ \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial w_0}{\partial \theta} \right\}$$

$$\frac{\partial^2 w}{\partial x^2} = \cos^2\theta \frac{\partial^2 w}{\partial r^2} - \frac{\sin 2\theta}{r}\frac{\partial^2 w}{\partial r\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial w}{\partial r} + \frac{\sin 2\theta}{r^2}\frac{\partial w}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$

$$\frac{\partial^2 w}{\partial y^2} = \sin^2\theta \frac{\partial^2 w}{\partial r^2} + \frac{\sin 2\theta}{r}\frac{\partial^2 w}{\partial r\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial w}{\partial r} - \frac{\sin 2\theta}{r^2}\frac{\partial w}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial \phi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\sin\theta\right)$$

$$= \sin\theta\cos\theta \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\sin^2\theta \frac{\partial \phi}{\partial \theta} - \frac{\sin^2\theta}{r}\frac{\partial^2 \phi}{\partial r\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial \phi}{\partial r}$$

$$- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\cos^2\theta}{r^2}\frac{\partial \phi}{\partial \theta}$$

$$= \sin\theta\cos\theta \frac{\partial^2 \phi}{\partial r^2} - \frac{\cos 2\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\cos 2\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial \phi}{\partial r}$$

$$- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

Thus

$$\left(\frac{\partial^2\omega}{\partial x\partial y}\right)^2 = \sin^2\theta\cos^2\theta\left(\frac{\partial^2\varphi}{\partial r^2}\right)^2 + \frac{\cos^2 2\theta}{r^4}\left(\frac{\partial\varphi}{\partial\theta}\right)^2 + \frac{\cos^2 2\theta}{r^2}\left(\frac{\partial^2\varphi}{\partial r\partial\theta}\right)^2$$

$$+ \frac{\sin^2\theta\cos^2\theta}{r^2}\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{\sin^2\theta\cos^2\theta}{r^4}\left(\frac{\partial^2\varphi}{\partial\theta^2}\right)^2 - \frac{\sin 2\theta\cos 2\theta}{r^2}\frac{\partial\varphi}{\partial\theta}\frac{\partial^2\varphi}{\partial r^2}$$

$$+ \frac{\sin 2\theta\cos 2\theta}{r}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{1}{2}\frac{\sin^2 2\theta}{r}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2} - \frac{1}{2}\frac{\sin^2 2\theta}{r^2}\frac{\partial^2\varphi}{\partial r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$- 2\frac{\cos^2 2\theta}{r^3}\frac{\partial\varphi}{\partial\theta}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{\sin 2\theta\cos 2\theta}{r^3}\frac{\partial\varphi}{\partial r}\frac{\partial\varphi}{\partial\theta} + \frac{\sin 2\theta\cos 2\theta}{r^4}\frac{\partial\varphi}{\partial\theta}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$- \frac{\sin 2\theta\cos 2\theta}{r^2}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{\sin 2\theta\cos 2\theta}{r^3}\frac{\partial^2\varphi}{\partial r\partial\theta}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{2}\frac{\sin^2 2\theta}{r^3}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\boxed{\left(\frac{\partial^2\omega}{\partial x\partial y}\right)^2 - \frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial y^2} = \frac{1}{r^4}\left(\frac{\partial\omega}{\partial\theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial^2\omega}{\partial r\partial\theta}\right)^2 - \frac{1}{r}\frac{\partial\omega}{\partial r}\frac{\partial^2\omega}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\omega}{\partial r^2}\frac{\partial^2\omega}{\partial\theta^2} - 2\frac{1}{r^3}\frac{\partial\omega}{\partial\theta}\frac{\partial^2\omega}{\partial r\partial\theta}}$$

We have
$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$
$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left\{\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2}{2}\right\}$$

Thus

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial \theta} = -\left(\frac{r}{R}\right)^2 \sin\theta\cos\theta = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial w_0}{\partial \theta} = -\left(\frac{r}{R}\right)^2 \sin\theta\cos\theta = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r\partial\theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r\partial\theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial\theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial\theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta + f\frac{1}{r}\left(\frac{r}{R}\right)^2 \\ \frac{1}{R}\frac{\partial w_0}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta + \frac{f}{R^2} \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta \end{cases}$$

$$\frac{1}{r^4}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{1}{r^4}\left(\frac{\partial w_0}{\partial \theta}\right)^2 = 0$$

$$\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial r \partial \theta}\right)^2 - \frac{1}{r^2}\left(\frac{\partial^2 w_0}{\partial r \partial \theta}\right)^2 = 0$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\} = \left(\frac{1}{R^2}\sin^2\theta\right)^2 - \left[\frac{1}{R^2}\sin^2\theta - \frac{f}{R^2}\right]^2$$

$$= \frac{1}{R^4}\left[2\sin^2\theta - f\right]f \cdot R^2$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\} = \frac{1}{R^2}\cos 2\theta\left\{\frac{f}{R^2}\right\} = \frac{1}{R^4} f \cos 2\theta \cdot R^2$$

$$-2\left\{\frac{1}{r^3}\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial r \partial \theta} \ldots\right\} = \frac{1}{R^2}\sin 2\theta\left\{\cdot\ \cdot\right\} = 0$$

The equation for the stress function φ is simply

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) = \frac{Ef}{R^2}\left\{\cos 2\theta + 2\sin^2\theta - f\right\}$$

$$= \frac{Ef(1-f)}{R^2} = K$$

First we have to find the particular integral of the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) = K$$

Assuming φ independent of θ,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) = K$$

Let $\varphi = c r^p$

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} = [p(p-1) + p] r^{p-2} = c p^2 r^{p-2}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) = c p^2 (p-2)^2 r^{p-4} = K$$

$\therefore \quad \underline{\underline{p = 4}}$

Also $\quad \varphi = c r^p$

$$c \cdot p^2 (p-2)^2 = K \qquad c = \frac{K}{16 \cdot 4} = \underline{\underline{\frac{K}{64}}}$$

Hence the particular integral is

$$\boxed{\varphi = \frac{K}{64} r^4}$$

Due to the symmetry of this problem, the solution of the homogeneous equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) = 0$$

can be written as

$$\varphi = \sum_{n=1}^{\infty} r^{2n}\left(A_n \cos 2(n-1)\theta + B_n \cos 2n\theta\right) \quad + \Sigma \cos 2\theta)$$

Therefore the complete solution is

$$\varphi = \frac{K}{64} r^4 + \sum_{n=1}^{\infty} r^{2n}\left[A_n \cos 2(n-1)\theta + B_n \cos 2n\theta\right] \cdots \qquad \cdots 4)$$

Thus

$$\sigma_r = \frac{K}{16} r^2 + \sum_1 r^{2(n-1)} \left[A_n \{2n - 4(n-1)^2\} \cos 2(n-1)\theta + B_n \{2n - 4n^2\} \cos 2n\theta \right]$$

$$\sigma_\theta = \frac{3K}{16} r^2 + \sum_1 2n(2n-1) r^{2(n-1)} \left[A_n \cos 2(n-1)\theta + B_n \cos 2n\theta \right]$$

$$\tau_{r\theta} = \sum_1 (2n-1) r^{2(n-1)} \left[2(n-1) A_n \sin 2(n-1)\theta + 2n B_n \sin 2n\theta \right]$$

The boundary conditions are at $r = a$, $\sigma_r = \tau_{r\theta} = 0$, $\sigma_\theta = \sigma(1 - 2\cos 2\theta)$

Thus

$$0 = \frac{K}{16} a^2 + \sum_1 \left[\{2n - 4(n-1)^2\} a_n \cos 2(n-1)\theta + \{2n - 4n^2\} b_n \cos 2n\theta \right]$$

$$\sigma(1 - 2\cos 2\theta) = \frac{3K}{16} a^2 + \sum_1 2n(2n-1) \left[a_n \cos 2(n-1)\theta + b_n \cos 2n\theta \right]$$

$$0 = \sum_1 (2n-1) \left[2(n-1) a_n \sin 2(n-1)\theta + 2n b_n \sin 2n\theta \right]$$

where $a_n = a^{2(n-1)} A_n$

$$\therefore \quad \frac{Ef(1-f)}{16} \left(\frac{a}{R}\right)^2 = -2a_1 , \quad 0 = \{2(n+1) - 4n^2\} a_{n+1} + \{2n - 4n^2\} b_n$$

$$\sigma - \frac{3Ef(1-f)}{16} \left(\frac{a}{R}\right)^2 = 2a_1 , \quad -2\sigma = 2b_1 + 12a_2 ,$$

$$2(n+1)[2n+1] a_{n+1} + 2n(2n-1) b_n = 0 .$$

$$(2n+1)2n\, a_{n+1} + 2n(2n-1)\, b_n = 0$$

<u>It is thus impossible to satisfy the stress boundary conditions.</u>

But

$$\sigma_r = E\frac{\partial u}{\partial r} + \frac{E}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 - \frac{\partial w_0}{\partial r}\ldots\right\}$$

$$= E\frac{\partial u}{\partial r} + \frac{E}{2} f\left(\frac{r}{R}\right)\left[f\left(\frac{r}{R}\right) - 2\left(\frac{r}{R}\right)\sin^2\theta\right]$$

$$= E\frac{\partial u}{\partial r} + \frac{E}{2} f(f-1)\left(\frac{r}{R}\right)^2 + \frac{E}{2} f\left(\frac{r}{R}\right)^2 \cos 2\theta$$

$$\sigma_\theta = E\left\{\frac{u}{r} + \frac{\partial v}{r\partial\theta}\right\}$$

$$\tau_{r\theta} = \frac{E}{2}\left\{\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right\} - \frac{E}{2}\frac{1}{2}\left(\frac{r}{R}\right)\sin 2\theta \cdot f\left(\frac{r}{R}\right)$$

Hence

$$E\frac{\partial u}{\partial r} = \frac{9}{16} E f(1-f)\left(\frac{r}{R}\right)^2 - \frac{1}{2} E f\left(\frac{r}{R}\right)^2 \cos 2\theta$$

$$+ \sum \left(\frac{r}{R}\right)^{2(n-1)}\Big[A_n\left\{2n - 4(n-1)^2\right\}\cos 2(n-1)\theta$$

$$+ B_n\left\{2n - 4n^2\right\}\cos 2n\theta\Big]$$

$$E\left[\frac{u}{r} + \frac{\partial v}{r\partial\theta}\right] = \frac{3Ef(1-f)}{16}\left(\frac{r}{R}\right)^2 + \sum 2n(2n-1)\left(\frac{r}{R}\right)^{2(n-1)}\Big[A_n\cos 2(n-1)\theta$$

$$+ B_n \cos 2n\theta\Big]$$

$$E\left[\frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right] = \frac{1}{2} f\left(\frac{r}{R}\right)^2 \sin 2\theta$$

$$+ \sum 2(2n-1)\left(\frac{r}{R}\right)^{2(n-1)}\Big[2(n-1)A_n \sin 2(n-1)\theta$$

$$+ 2n B_n \sin 2n\theta\Big]$$

$$E\frac{u}{R} = \frac{3}{16}Ef(1-f)\left(\frac{r}{R}\right)^3 - \frac{1}{6}Ef\left(\frac{r}{R}\right)^3\cos 2\theta$$

$$+ \sum \frac{1}{2n-1}\left(\frac{r}{R}\right)^{2n-1}\left[A_n\left\{2n-4(n-1)^2\right\}\cos 2(n-1)\theta + B_n\left\{2n-4n^2\right\}\cos 2n\theta\right]$$

$$E\frac{u}{R} = \frac{3}{16}Ef(1-f)\left(\frac{r}{R}\right)^3 - \frac{1}{6}Ef\left(\frac{r}{R}\right)^3\cos 2\theta$$

$$+ \sum\left(\frac{r}{R}\right)^{2n-1}\left[A_n\,(4-2n)\cos 2(n-1)\theta - B_n\,2n\cos 2n\theta\right]$$

$$E\frac{u}{r} = \frac{3}{16}Ef(1-f)\left(\frac{r}{R}\right)^2 - \frac{1}{6}Ef\left(\frac{r}{R}\right)^2\cos 2\theta$$

$$+ \sum\left(\frac{r}{R}\right)^{2(n-1)}\left[(4-2n)A_n\cos 2(n-1)\theta - 2nB_n\cos 2n\theta\right]$$

$$E\frac{1}{r}\frac{\partial v}{\partial\theta} = \frac{1}{6}Ef\left(\frac{r}{R}\right)^2\cos 2\theta + \sum\left(\frac{r}{R}\right)^{2(n-1)}\left[4(n^2-1)A_n\cos 2(n-1)\theta + 4n^2B_n\cos 2n\theta\right]$$

$$E\frac{v}{R} = \frac{1}{12}Ef\left(\frac{r}{R}\right)^3\sin 2\theta + \sum\left(\frac{r}{R}\right)^{2n-1}\left[-2(n+1)A_n\sin 2(n-1)\theta + 2n\,B_n\sin 2n\theta\right]$$

But from page 267

$$E\frac{u}{R} = \sigma\left(\frac{a}{R}\right)\left[1+2\cos 2\theta\right]$$

$$E\frac{v}{R} = -\sigma\left(\frac{a}{R}\right)\sin 2\theta$$

$$\sigma \frac{a}{R} = \frac{3}{16} E\, f(1-f)\left(\frac{a}{R}\right)^3 + \left(\frac{a}{R}\right)\cdot 2A_1$$

$$2\sigma\left(\frac{a}{R}\right) = -\frac{1}{6} E f \left(\frac{a}{R}\right)^3 - 2\left(\frac{a}{R}\right) B_1$$

$$-\sigma\left(\frac{a}{R}\right) = \frac{1}{12} E f \left(\frac{a}{R}\right)^3 + 2\left(\frac{a}{R}\right) B_1$$

Three equations for three knowns f, A_1, B_1

$$\frac{\sigma}{E} = -\frac{1}{12} f \left(\frac{a}{R}\right)^2 \qquad \therefore \qquad \boxed{f = \frac{-12\sigma}{E\left(\frac{a}{R}\right)^2}}$$

$$b_1 = -\frac{1}{6} E f \left(\frac{a}{R}\right)^2 - \sigma$$

$$= 2\sigma - \sigma = \sigma. \qquad \boxed{B_1 = \sigma}$$

$$\therefore \qquad \sigma = \frac{3}{16} E f (1-f)\left(\frac{a}{R}\right)^2 + 2A_1$$

$$A_1 = \frac{\sigma}{2} - \frac{3}{32}(-12\sigma)\left[1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right]$$

$$\boxed{A_1 = \frac{\sigma}{2} + \frac{9}{8}\sigma\left[1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right]}$$

$$\left(\frac{\varphi}{R^2}\right) = \underbrace{\frac{Ef(1-f)}{64}}_{C_1}\left(\frac{r}{R}\right)^4 + \left(\frac{r}{R}\right)^2 \underbrace{\left[\frac{\sigma}{2} - \frac{3}{32}Ef(1-f)\left(\frac{a}{R}\right)^2\right]}_{C_2}$$

$$+ \left(\frac{r}{R}\right)^2 \sigma \cos 2\theta \qquad C_3$$

or $$\frac{\varphi}{R^2} = C_1\left(\frac{r}{R}\right)^4 + C_2\left(\frac{r}{R}\right)^2 + C_3\left(\frac{r}{R}\right)^2 \cos 2\theta$$

$$\sigma_r = 4C_1\left(\frac{r}{R}\right)^2 + 2C_2 + 2C_3 \cos 2\theta - 4C_3 \cos 2\theta$$

$$\sigma_\theta = 12C_1\left(\frac{r}{R}\right)^2 + 2C_2 + 2C_3 \cos 2\theta$$

$$\tau_{r\theta} = + 2C_3 \sin 2\theta$$

the extensional energy

$$4\,\frac{Et}{2}\,\frac{1}{E^2}\,R^2 \int_0^{a/R}\int_0^{2\pi} \Big\{\left[2C_1\left(\frac{r}{R}\right)^2 + C_2 - C_3\cos 2\theta\right]^2$$

$$+ \left[6C_1\left(\frac{r}{R}\right)^2 + C_2 + C_3\cos 2\theta\right]^2 + 2C_3^2\sin^2 2\theta\Big\}\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) d\theta$$

$$= \frac{2tR^2\pi}{E}\int_0^{a/R}\left[8C_1^2\left(\frac{r}{R}\right)^4 + 2C_2^2 + C_3^2 + 72C_1^2\left(\frac{r}{R}\right)^4 + 2C_2^2 + C_3^2 + 2C_3^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

$$= \frac{2tR^2\pi}{E}\int_0^{a/R}\left[80C_1^2\left(\frac{r}{R}\right)^5 + 4\left(C_2^2 + C_3^2\right)\frac{r}{R}\right] d\left(\frac{r}{R}\right)$$

$$= \frac{8tR^2\pi}{E}\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^6 + \frac{1}{2}\left(C_2^2 + C_3^2\right)\left(\frac{a}{R}\right)^2\right\}$$

The bending energy

~~$$\frac{1}{R}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] - \frac{1}{R}\left[\frac{\partial^2 w_0}{\partial R^2} + \frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{\mu}{r^2}\frac{\partial w_0}{\partial \theta^2}\right]$$~~

~~$$= \frac{1}{R^2}\Big[$$~~

In case of polar coordinates

$$k_1 = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 w_0}{\partial r^2}$$

$$k_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}$$

$$\tau = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_0}{\partial \theta}\right)$$

$$\frac{1}{R}k_1 = \frac{f}{R^2}$$

$$\tau = 0$$

$$\frac{1}{R}k_2 = \frac{f}{R^2}$$

$$\text{Bending energy} = \frac{Et}{2}\frac{t^2}{12}\,2\left(\frac{f}{R}\right)^2\pi a^2 = \frac{E}{12}\left(\frac{t}{R}\right)^3\pi\left(\frac{a}{R}\right)^2 f^2 R^3$$

Total energy

$$\frac{W}{R^3} = \frac{8\left(\frac{t}{R}\right)}{E}\pi\left(\frac{a}{R}\right)^2\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4 + \frac{1}{2}(C_2^2 + C_3^2)\right\} + \frac{E}{12}\left(\frac{t}{R}\right)^3 f^2\,\pi\left(\frac{a}{R}\right)^2$$

$$= \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\left[\frac{8}{E}\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4 + \frac{1}{2}(C_2^2 + C_3^2)\right\} + \frac{E}{12}\left(\frac{t}{R}\right)^2 f^2\right]$$

Decrease in energy

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\left[\frac{\sigma^2}{2E} - \frac{8}{E}\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4 + \frac{1}{2}(C_2^2 + C_3^2)\right\} - \frac{E}{12}\left(\frac{t}{R}\right)^2 f^2\right]$$

Total decrease in energy

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\left[\frac{9}{32}\frac{\sigma^2}{E} - \frac{8}{E}\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4 + \frac{1}{2}(C_2^2 + C_3^2)\right\} - \frac{E}{12}\left(\frac{t}{R}\right)^2 f^2\right]$$

Or write out completely, we have

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\left[\frac{9}{32}\frac{\sigma^2}{E} - \frac{8}{E}\left\{\frac{15}{128}\sigma^2\left[1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right]^2 + \frac{1}{2}\left[\sigma^2 + \left\{\frac{\sigma}{2} + \frac{9}{8}\sigma\left(1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right)\right\}^2\right]\right\} - \frac{E}{12}\left(\frac{t}{R}\right)^2\frac{144\sigma^2}{E^2\left(\frac{a}{R}\right)^4}\right]$$

$$= \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{E}\left[\frac{9}{32} - \frac{15}{16}\left\{1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right\}^2 - 5 - \frac{81}{16}\left\{1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right\}^2 - \frac{9}{2}\left\{1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right\} - 12\left(\frac{t}{R}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4}\right]$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{E}\left[\frac{9}{32} - 5 - 6\left\{1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right\}^2 - \frac{9}{2}\left\{1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right\} - 12\left(\frac{t}{R}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4}\right]$$

$$= \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{E}\left[-\frac{487}{32} - 198\frac{\sigma}{E\left(\frac{a}{R}\right)^2} - 864\left(\frac{\sigma}{E}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4} - 12\left(\frac{t}{R}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4}\right]$$

Putting $\sigma = -\sigma$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left[198\left(\frac{\sigma}{E}\right) - \frac{487}{32}\left(\frac{a}{R}\right)^2 - 864\left(\frac{\sigma}{E}\right)^2\frac{1}{\left(\frac{a}{R}\right)^2} - 12\left(\frac{\sigma}{E}\right)^2\left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\frac{1}{\left(\frac{a}{R}\right)^2}\right]$$

$$\frac{487}{32}\left(\frac{a}{R}\right)^4 = 12\left(\frac{\sigma}{E}\right)^2\left[72 + \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\right]$$

$$\therefore \left(\frac{a}{R}\right)^2 = \sqrt{\frac{384}{487}}\,\frac{\sigma}{E}\left[72 + \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\right]^{\frac{1}{2}}$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left[198\left(\frac{\sigma}{E}\right) - 2\sqrt{\frac{487\times 12}{32}}\left(\frac{\sigma}{E}\right)\left\{72 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2\right\}^{\frac{1}{2}}\right]$$

$$= \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left(\frac{\sigma}{E}\right)\left[198 - 2\sqrt{\frac{1461}{8}\left\{72 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2\right\}}\right]$$

Thus at σ_c, $\Delta W = 0$

$$198^2 = \frac{1461}{8}\left\{72 + \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\right\}$$

$$\sqrt{198^2 \frac{8}{1461} - 66} = \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right) = 12.19$$

100

300000

214.66
66
148.66

$$\boxed{\sigma_{cri} = 0.0821\, E\left(\frac{t}{R}\right)}$$

at σ_{cri}

$$198\sqrt{\frac{32}{487 \times 12}} = \sqrt{66 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2}$$

$$\left(\frac{a}{R}\right)^2 = \frac{32}{487}\,\frac{\sigma}{E} = \frac{32}{487} \times 0.0821\left(\frac{t}{R}\right)$$

$$\boxed{\left(\frac{a}{R}\right) = 0.0735\sqrt{\left(\frac{t}{R}\right)}}$$

The residual stress at the edges are

$$\sigma_r = 4C_1\left(\frac{a}{R}\right)^2 + 2C_2 - 2C_3\cos 2\theta$$

$$\sigma_r = \frac{Ef(1-f)}{16}\left(\frac{a}{R}\right)^2 + \left\{\sigma - \frac{3}{16}Ef(1-f)\left(\frac{a}{R}\right)^2\right\} - 2\sigma\cos 2\theta$$

$$\sigma_\theta = \frac{3Ef(1-f)}{16}\left(\frac{a}{R}\right)^2 + \left\{\sigma - \frac{3}{16}Ef(1-f)\left(\frac{a}{R}\right)^2\right\} + 2\sigma\cos 2\theta$$

$$\tau_{r\theta} = 2\sigma\sin 2\theta$$

The additional stress system is

$$\begin{cases} \sigma_r = -\frac{Ef(1-f)}{8}\left(\frac{a}{R}\right)^2 + \sigma(1 - 2\cos 2\theta) \\ \sigma_\theta = 4\sigma\cos 2\theta \\ \tau_{r\theta} = 2\sigma\sin 2\theta \end{cases}$$

[See p. 385, Southwell's "elasticity", Eq. (43) & (45)

$$R_0 = \sigma - \frac{Ef(1-f)}{8}\left(\frac{a}{R}\right)^2$$

$$\left.\begin{aligned} 2\sigma &= 6Q_2 + 4S_2 \\ -2\sigma &= 6Q_2 + 2S_2 \end{aligned}\right\} \quad 4\sigma = 2S_2 \qquad S_2 = 2\sigma, \quad Q_2 = -\sigma$$

Notice above that only the σ_r & $\tau_{r\theta}$ conditions are satisfied

$$\sigma_r = \left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}\left(\frac{a}{r}\right)^2 + \left\{6\sigma\left(\frac{a}{r}\right)^4 - 8\sigma\left(\frac{a}{r}\right)^2\right\}\cos 2\theta$$

$$\sigma_\theta = -\left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}\left(\frac{a}{r}\right)^2 - 6\sigma\left(\frac{a}{r}\right)^4 \cos 2\theta$$

$$\tau_{r\theta} = \left\{6\sigma\left(\frac{a}{r}\right)^4 - 4\sigma\left(\frac{a}{r}\right)^2\right\}\sin 2\theta$$

$$\pi a^2 \frac{t}{2E}\int_1^\infty \Bigg[2\left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}^2 \frac{1}{\left(\frac{r}{a}\right)^3} + \left\{6\sigma\frac{1}{\left(\frac{r}{a}\right)^4} - 8\sigma\frac{1}{\left(\frac{r}{a}\right)^2}\right\}^2\left(\frac{r}{a}\right)$$

$$+ 2\left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}^2\frac{1}{\left(\frac{r}{a}\right)^3} + 36\sigma^2\frac{1}{\left(\frac{r}{a}\right)^7}$$

$$+ 2\left\{6\sigma\frac{1}{\left(\frac{r}{a}\right)^4} - 4\sigma\frac{1}{\left(\frac{r}{a}\right)^2}\right\}^2\left(\frac{r}{a}\right)\Bigg]\, d\left(\frac{r}{a}\right)$$

$$= \frac{\pi a^2 t\sigma^2}{2E}\left[2\left\{1 - \frac{E f(1-f)}{\sigma 8}\left(\frac{a}{R}\right)^2\right\}^2 + (6 - 24 + 32 + 6 + 12 - 24 + 16)\right]$$

$$= \frac{\pi a^2 t\sigma^2}{E}\left\{\left[1 - \frac{E f(1-f)}{8\sigma}\left(\frac{a}{R}\right)^2\right]^2 + 12\right\}$$

$$\frac{\Delta W_1}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{13\left(\frac{a}{R}\right)^2 - \frac{Ef(1-f)}{4\sigma}\left(\frac{a}{R}\right)^4 + \frac{E^2f^2(1-f)^2}{64\sigma^2}\left(\frac{a}{R}\right)^6\right\}$$

$$= \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{13\left(\frac{a}{R}\right)^2 + 3\left(1+\frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right)\left(\frac{a}{R}\right)^4 + \frac{9}{4}\left(1+\frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right)^2\left(\frac{a}{R}\right)^6\right\}$$

$$= \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{13\left(\frac{a}{R}\right)^2 + 3\left(\frac{a}{R}\right)^4 + \frac{9}{4}\left(\frac{a}{R}\right)^6 + \frac{36\sigma}{E}\left(\frac{a}{R}\right)^2 + \frac{54\sigma}{E}\left(\frac{a}{R}\right)^4 + 9\times 36\left(\frac{\sigma}{E}\right)^2\left(\frac{a}{R}\right)^2\right\}$$

$$\frac{\Delta W_1}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{\left[13 - 36\left(\frac{\sigma}{E}\right) + 324\left(\frac{\sigma}{E}\right)^2\right]\left(\frac{a}{R}\right)^2 + \left[3 - 54\frac{\sigma}{E}\right]\left(\frac{a}{R}\right)^4 + \frac{9}{4}\left(\frac{a}{R}\right)^6\right\}$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{198\left(\frac{\sigma}{E}\right) - \left[\frac{903}{32} - 36\left(\frac{\sigma}{E}\right) + 324\left(\frac{\sigma}{E}\right)^2\right]\left(\frac{a}{R}\right)^2 - \left[3 - 54\left(\frac{\sigma}{E}\right)\right]\left(\frac{a}{R}\right)^4 - \frac{9}{4}\left(\frac{a}{R}\right)^6 - 12\left(\frac{\sigma}{E}\right)^2\left[72 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2\right]\frac{1}{\left(\frac{a}{R}\right)^2}\right\}$$

Recalculation of Strain energy increase due to the presence of a hole:

$$\widehat{rr} = \frac{1}{2}\sigma\left(1-\frac{a^2}{r^2}\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]$$

$$\widehat{\theta\theta} = \frac{1}{2}\sigma\left[\left(1+\frac{a^2}{r^2}\right) - \left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$\widehat{r\theta} = -\frac{1}{2}\sigma\left(1-\frac{a^2}{r^2}\right)\left(1+3\frac{a^2}{r^2}\right)\sin 2\theta$$

$$\widehat{rr}_0 = \frac{1}{2}\sigma\left(1+\cos 2\theta\right)$$

$$\widehat{\theta\theta}_0 = \frac{1}{2}\sigma\left(1-\cos 2\theta\right)$$

$$\widehat{r\theta}_0 = -\frac{1}{2}\sigma\sin 2\theta$$

Strain energy increase due to the presence of a hole

$$= \frac{\pi a^2 t\sigma^2}{8E}\int_1^\infty \left(\frac{r}{a}\right)d\left(\frac{r}{a}\right)\left[\left(1-\frac{a^2}{r^2}\right)^2\left\{2+\left(1-3\frac{a^2}{r^2}\right)^2\right\} + 2\left(1+\frac{a^2}{r^2}\right)^2 + \left(1+\frac{3a^4}{r^4}\right)^2 + 2\left(1-\frac{a^2}{r^2}\right)^2\left(1+3\frac{a^2}{r^2}\right)^2 - 8\right]$$

$$= \frac{\pi a^2 t\sigma^2}{8E}\int_1^\infty \left(\frac{r}{a}\right)d\left(\frac{r}{a}\right)\left[\left(1-2\frac{a^2}{r^2}+\frac{a^4}{r^4}\right)\left(3-6\frac{a^2}{r^2}+9\frac{a^4}{r^4}\right) + 2\left(1+2\frac{a^2}{r^2}+\frac{a^4}{r^4}\right) + \left(1+6\frac{a^4}{r^4}+9\frac{a^8}{r^8}\right) + 2\left(1+4\frac{a^2}{r^2}-2\frac{a^4}{r^4}-12\frac{a^6}{r^6}+9\frac{a^8}{r^8}\right) - 8\right]$$

$$= \frac{\pi a^2 t\sigma^2}{2E}\int_1^\infty \left(\frac{r}{a}\right)d\left(\frac{r}{a}\right)\left[7\left(\frac{a}{r}\right)^4 - 12\left(\frac{a}{r}\right)^6 + 9\left(\frac{a}{r}\right)^8\right] = \frac{\pi a^2 t\sigma^2}{2E}\left[\frac{1}{2}+\frac{9}{6}\right]$$

$$\therefore \boxed{\frac{\Delta W_1}{R^3} = \pi \left(\frac{a}{R}\right)^2 \left(\frac{t}{R}\right) \frac{\sigma^2}{E}}$$

Refering to page 81 Impossible !!!

［Ｇｅｎｅｒａｌ Ｉｎｆｏｒｍａｔｉｏｎ］
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